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Professor R. A. Sampson, M.A., D.Sc., F.R.S., PRESIDENT, in the Chair.

The Hon. Sir Charles Algernon Parsons, K.C.B., M.A., D.Sc., LL.D., F.R.S., 6 Windsor Terrace, Newcastle-on-Tyne; and

Manning Robertson, A.R.I.B.A., 50 Norfolk Square, London, W.,

were balloted for and duly elected Fellows of the Society.

The following candidates were proposed for election as Fellows of the Society, the names of the proposers from personal knowledge being appended:—

John Beebe, "La Rocca," East Bendigo, Victoria, Australia (proposed by C. J. Merfield);

George Rusby Kaye, Curator of the Bureau of Education of the Government of India, Simla, India (proposed by H. G. Tomkins); and

John Stewart, Raleigh Street 45, Essendon, Melbourne, Australia (proposed by C. J. Merfield).

The President announced that the Council had awarded the Gold Medal of the Society to Mr. Walter Sydney Adams, of the Mount Wilson Observatory, for his investigations in Stellar Spectroscopy, including the Determination of absolute magnitudes.

Fifty presents were announced as having been received since the last meeting, including, amongst others:—

Vol. I. of the Annals of the Dearborn Observatory, Evanston, Illinois (Measures of Double Stars), presented by the Observatory.

The Part played by Rotation in Cosmic Evolution.

By J. H. Jeans, M.A., F.R.S.

1. The present paper contains a condensed account of some investigations on the rotational theory of cosmic evolution. Two ways are known in which a mass can be broken up under increasing rotation. In the first way (Poincaré-Darwin) the mass first flattens, then loses its symmetry, becoming first ellipsoidal and then pear-shaped, and then throws off a satellite. In the second way (Laplace-Roche) the flattened stage is directly followed by the shedding of a ring of matter at the equator, and this ring may break up into satellites. The results obtained in the present paper indicate that these are representative (although very special) cases of two distinct general methods of break-up. These two general methods are found to be the only possible methods of break-up, and neither of them seems to be able to explain the genesis of our solar system. The first method of break-up appears to lead to the normal binary star formation, while the second seems likely to lead to the spiral nebula formation. Under this second method of break-up it is found that the ejected matter will only condense into detached masses when the parent mass is much greater than our Sun.

Mathematical Preliminaries.

2. The equations of equilibrium for any mass in rotation with a uniform angular velocity ω about the axis of z are

$$\frac{\partial p}{\partial x} = \rho \frac{\partial V}{\partial x} + \omega^2 \rho x = \rho \frac{\partial \Omega}{\partial x} \quad . \quad . \quad . \quad (1)$$

and two similar equations, where

$$\Omega = V + \tfrac{1}{2}\omega^2(x^2 + y^2) \quad . \quad . \quad . \quad . \quad (2)$$

In these equations V denotes the total gravitational potential including that of tidal forces, if any are present. We may put

$$V = V_M + V_T,$$

when V_M is the gravitational potential of the rotating mass, and V_T is the potential of any tidal forces. The condition that a figure shall be one of equilibrium for a mass rotating freely in space is that V_T shall vanish.

Let P be any point inside the mass; let R be the distance

from P to a point x, y, z, also inside the mass; let dS' be an element of surface of a small sphere surrounding P, and let dS be an element of surface of the boundary. Then since $\nabla^2 V = -4\pi\rho$, we have

$$\begin{aligned} V_M &= -\frac{1}{4\pi}\iiint \frac{\nabla^2 V}{R}\,dx\,dy\,dz \\ &= \frac{1}{4\pi}\iint\left\{\frac{\partial V}{\partial n}\left(\frac{1}{R}\right) - V\frac{\partial}{\partial n}\left(\frac{1}{R}\right)\right\}(dS + dS') \\ &= V + \frac{1}{4\pi}\iint\left\{\frac{\partial V}{\partial n}\left(\frac{1}{R}\right) - V\frac{\partial}{\partial n}\left(\frac{1}{R}\right)\right\}dS. \end{aligned}$$

If the conditions of equilibrium (1) are satisfied, Ω is constant over the boundary, so that on the boundary

$$V = C - \tfrac{1}{2}\omega^2(x^2 + y^2),$$

where C is a constant, and hence we obtain, as the value of V_T at the point P,

$$\begin{aligned} V_T = V - V_M &= -\frac{1}{4\pi}\iint\left\{\frac{\partial V}{\partial n}\left(\frac{1}{R}\right) - V\frac{\partial}{\partial n}\left(\frac{1}{R}\right)\right\}dS \\ &= -\frac{1}{4\pi}\iint\frac{\partial V}{\partial n}\left(\frac{1}{R}\right)dS + \tfrac{1}{2}\omega^2\iint(x^2+y^2)\frac{\partial}{\partial n}\left(\frac{1}{R}\right)dS, \end{aligned}$$

omitting a constant which has no physical importance. It appears that if V_T, ω, and the shape of the boundary are given, then

$$\iint\frac{\partial V}{\partial n}\left(\frac{1}{R}\right)dS$$

is determined at every point P inside the mass. Hence $\partial V/\partial n$ is determined at every point of the boundary, so that of course $\partial\Omega/\partial n$ is determined at every point of the boundary.

3. From equations (1) we readily obtain

$$\frac{\partial\rho}{\partial y}\frac{\partial\Omega}{\partial x} = \frac{\partial\rho}{\partial x}\frac{\partial\Omega}{\partial y}, \text{etc.},$$

whence it readily follows that the surfaces $\Omega = \text{cons.}$ coincide with the surfaces of constant density, and therefore also, by equation (1), with the surfaces of constant pressure.

We also obtain from equations (1)

$$\frac{\partial}{\partial x}\left(\frac{1}{\rho}\frac{\partial p}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{\rho}\frac{\partial p}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{1}{\rho}\frac{\partial p}{\partial z}\right) = \nabla^2\Omega = 2\omega^2 - 4\pi\rho \qquad (3)$$

Multiply both sides of the equation by $dx\,dy\,dz$ and integrate throughout any closed surface inside the matter and we obtain

$$\left.\begin{aligned} -\iint\frac{1}{\rho}\frac{\partial p}{\partial n}dS &= \iiint(2\omega^2 - 4\pi\rho)dx\,dy\,dz \\ &= 2\omega^2 W - 4\pi M \end{aligned}\right\} \qquad (4)$$

where W is the volume and M the mass enclosed within the surface.*

Compressible Masses.

4. The material now obtained puts us in a position to discuss the configurations of equilibrium of compressible and non-homogeneous masses. To separate the difficulties, let us first consider masses of compressible matter which are homogeneous in structure, although not of course in density, so that the pressure is a function of the density. For such matter we may put

$$\int \frac{dp}{\rho} = \phi(\rho),$$

and equation (3) assumes the form

$$\nabla^2\phi(\rho) = 2\omega^2 - 4\pi\rho \quad . \quad . \quad . \quad . \quad (5)$$

the differential equation which must be satisfied by ρ for equilibrium to be possible. The boundary is determined by the condition $p = 0$, and therefore by a condition of the form $\rho = \sigma$, where σ is a constant which may be zero. For a special problem in which V_T and ω are given, and in which the boundary is also given, it follows from § 2 that ρ and $\partial\rho/\partial n$ are given at all points of the boundary, whence from equation (5) it follows that ρ can be uniquely determined at every point inside the mass. It follows that configurations of equilibrium may be classified by their boundaries.

When V_T and ω^2 are given for any problem, any given boundary configuration will permit of an infinite number of vibrations in which the internal configuration changes while the boundary does not. Our result shows that none of these vibrations can be of zero frequency, whence, from the general theory of points of bifurcation, it follows that all internal vibrations are necessarily stable.

For a freely rotating mass there must always be a series of configurations, analogous to the Maclaurin spheroids, in which the boundary is a figure of revolution. This series begins with the same configuration as the Maclaurin series, namely, a sphere; it also ends with the same configuration as the Maclaurin series, namely, a flat spheroidal disc of uniform density σ. Obviously the principal vibrations and stabilities of these end configurations must be the same on both series, so that the same points of bifurcation must occur on both. The general physical principle that vibrations of longest wave-length are the first to become unstable shows that the two series must both first lose their stability through displacements in which the circular cross-section gives place to an elliptic one. In this way we discover the existence of a series of what we may call pseudo-ellipsoidal configurations analogous to the Jacobian ellipsoids. This series begins in a figure of revolution on

* *Cf.* Poincaré, *Les Hypothèses Cosmogoniques*, p. 22.

the principal series; it ends in a long drawn-out filament which is identical with a Jacobian ellipsoid of uniform density σ. The stability of the end configuration on the pseudo-ellipsoidal series is the same as that of the end configuration on the Jacobian ellipsoid, whence, by the argument previously used, it follows that the first point of bifurcation on the pseudo-ellipsoidal series must lead to a pear-shaped deformation.

It does not appear to be possible to predict whether the pseudo-ellipsoidal series will, at its start, be stable or unstable; nor, if it is stable, is it possible to predict whether the pear-shaped series will be stable or unstable. But these questions are of minor importance. If the pseudo-ellipsoidal series is stable, the motion along it will be statical; if it is unstable, there will be dynamical motion either along it, or along configurations adjacent to it. In this latter case the pear-shaped point of bifurcation will be of the kind which may be described as a "dynamical point of bifurcation." * In any case the general sequence of changes in a compressible mass will be very similar to that in an incompressible mass, and appears likely to lead to a breaking up of the mass into two (or possibly more) detached bodies rotating round one another, as in the normal binary star formation.

Non-homogeneous Masses.

5. The discussion of the configurations of a non-homogeneous mass is naturally more difficult. Let the matter be of different types a, b, c . . ., these letters referring either to chemically distinct types of matter or to mixtures of such types in varying proportions.

Consider a special problem in which V_T, ω, and the shape of the boundary are given, and in which the shells of matter come in order a, b, c . . . from the boundary inwards. From § 3, it follows that surfaces of transition from one type of matter to another must be identical with surfaces $\Omega = \text{cons.}$ An equation such as (5) will be satisfied throughout each layer of matter, but will not be true at the interface between two layers.

As in § 4, ρ and dp/dn are determined at the boundary, so that, from equation (5), the arrangement of density is determined throughout the layer of matter of type a. Matter of type b comes next, and Ω and $\partial\Omega/\partial n$ must be continuous at the interface, so that ρ and $\partial\rho/\partial n$ are determined for matter of type b at this interface, and therefore through the whole layer. In this way we can build up the configuration layer by layer, and the configuration is unique when the order of the layers is determined. It follows that all internal vibrations are stable so long as the layers do not change places.

* *Cf.* a paper "On the Motion of Tidally distorted Masses," now in the press (*R.A.S. Memoirs*). If the pear-shaped point of bifurcation is dynamical, it appears to be possible for the mass to break into more than two detached pieces.

For given values of V_T and ω there will, however, be an infinite number of configurations corresponding to the different ways of arranging the layers. It must in general be supposed that convective movements in which the layers change places are physically possible, and when this is the case only one of all the possible configurations can be stable, namely, that in which the potential energy is least. But if convective movements are impossible, then every configuration is stable. In either case the argument can proceed as in § 4.

6. Two conditions must be satisfied by a configuration of equilibrium. One, which has already been discussed, is that the equations of equilibrium shall be satisfied; the other is, that p shall be positive everywhere. If p changes its sign at any point on a series of equilibrium configurations, the series must be supposed to be cut off abruptly at that point; configurations beyond, although satisfying the analytical conditions of equilibrium, are not physically possible.

It is easily seen that p cannot change sign at a point in the interior of the mass, for the preliminary condition for p to change sign is that p shall vanish. When p is just about to change sign we must have $\partial p/\partial n = 0$ at a point on the boundary. When this occurs, centrifugal and tidal forces outweigh the gravitational attraction at the point in question, and a stream of matter comes off.

The point at which $\partial p/\partial n$ vanishes must be a point of equilibrium on the equipotentials $\Omega = \text{cons.}$ Except when σ, the density at the boundary, vanishes, it is readily shown that the surface at such a point must be shaped like a sharp edge or a needle point. For instance, on the principal series of figures of revolution (analogous to the Maclaurin spheroids) $\partial p/\partial n$ cannot possibly vanish except on the edge of the limiting flat disc, while similarly on the pseudo-ellipsoidal series, $\partial p/\partial n$ cannot vanish except at the pointed ends of the limiting infinitely long ellipsoid. It is readily verified that $\partial p/\partial n$ does in actual fact vanish in both these instances.

When σ, the density at the boundary, vanishes, there is no such limitation on the points at which $\partial p/\partial n$ vanishes.

7. From the results just obtained it follows that for a solid or fluid or other medium in which the density does not vanish at the boundary, the birth of satellites can only take place through the first mechanism; the series of flattened figures of revolution gives place first to a series of pseudo-ellipsoidal figures, after which the satellite is thrown off.

For a gas either mechanism may come into play. On the principal series there are two points of importance,—P, the point of bifurcation at which the series loses its stability, and B, the breaking-up point. If B happens to be nearer to the spherical configuration than P, the second mechanism will come into operation, while if the converse is the case the figure of revolution will lose its symmetry after passing P, after which either mechanism

may come into operation.* But before a satellite can be born by rotation, the configuration must have passed through one or other of the points B and P.

At the point P we have

$$\frac{\omega^2}{2\pi\gamma\rho} = \cdot 18712 \quad . \quad . \quad . \quad . \quad (6)$$

when the mass is homogeneous and incompressible. Both compressibility and non-homogeneity appear to increase this value; † they permit of the more dense matter crowding near the centre where the centrifugal force is least, so that more rotation is required to produce a given effect. At the point B we have

$$\frac{\omega^2}{2\pi\gamma\bar{\rho}} = \cdot 36075 \quad . \quad . \quad . \quad (7)$$

when the mass is highly concentrated near its centre, $\bar{\rho}$ being the mean density M/W. When the mass is less concentrated the angular momentum appears to be greater than that determined by equation (7), and so *à fortiori* greater than that determined by equation (6).

Hence it appears almost certain that before a rotating mass can have thrown off a satellite it must have passed through a configuration of angular momentum greater than that defined by equation (6). This is not quite definitely proved, but the argument which follows will not be invalidated even if the number ·18712 is considerably reduced, while all available evidence goes to show that it ought to be increased.

The Solar System.

8. Let us now turn our attention specially to the solar system. For a body of mass M, moment of momentum M, and radius of gyration k, we have

$$\frac{\omega^2}{2\pi\gamma\bar{\rho}} = \frac{2M^2}{3\gamma \mathrm{M}^3} \frac{r_0^3}{k^4} \quad . \quad . \quad . \quad . \quad (8)$$

It appears that as a body shrinks, keeping its constitution the same, so that k^2/r_0^2 remains the same, $\omega^2/2\pi\gamma\bar{\rho}$ will *increase*, being proportional to $1/r_0$. The present value of this quantity for our Sun is

$$\frac{\omega^2}{2\pi\gamma\bar{\rho}} = \cdot 0000148 \quad . \quad . \quad . \quad . \quad (9)$$

the question before us is whether with shrinkage $\omega^2/2\pi\gamma\bar{\rho}$ can have *decreased* from the value (6) to the value (9).

* This at any rate represents the theoretical possibilities which are open. I have not succeeded in finding a case in which the second mechanism can operate when once the point of bifurcation P has been passed.

† *Phil. Trans.*, 213 A, p. 457.

Equation (8) shows that a decrease in $\omega^2/2\pi\gamma\bar{\rho}$ can be produced by an increase in M or by a decrease in M. Now M will doubtless have increased for our system, but a rough calculation shows that on no reasonable hypotheses can the increase in M have been enough to account for more than a very small fraction of the difference now before us. We shall accordingly assume M constant and equal to 2×10^{33} gms.

An interstellar atmosphere would decrease M, for the Sun would rotate in a viscous medium. The density of this medium is doubtless small, but then the viscosity of a gas is independent of its density. Calculations show that this agency would produce only an infinitesimal change in M in an interval of 10^{10} years. Tidal friction and other interactions between members of the solar system will decrease M, but an upper limit to this decrease is obtained, as Babinet pointed out,* by supposing the moment of momentum of the whole solar system all concentrated in the Sun. The value of this total moment of momentum is

$$M = 3{\cdot}3 \times 10^{50} \text{ C.G.S. units} \quad . \quad . \quad . \quad (10)$$

If this were all concentrated in the present Sun, regarded as homogeneous, the value of $\omega^2/2\pi\gamma\bar{\rho}$ would be increased to

$$\frac{\omega^2}{2\pi\gamma\bar{\rho}} = {\cdot}0117 \quad . \quad . \quad . \quad . \quad (11)$$

which is still far below the value given by equation (6).

9. Assigning to M its maximum possible value, as given by equation (10), the general equation (8) becomes

$$\frac{\omega^2}{2\pi\gamma\bar{\rho}} = 1{\cdot}34 \times 10^8 \frac{{r_0}^3}{k^4}.$$

Assuming the primæval nebula to have had a mean radius r_0 equal to n times the present radius of the Sun, it appears that equation (6) could only have been satisfied if k^2 were as small as $\cdot 101 n^{-\frac{1}{2}} {r_0}^2$. For instance, with r_0 equal to the radius of Neptune's orbit, the value of k^2 must have been $\cdot 0013 {r_0}^2$.

In this calculation we have certainly taken an excessive value for M. If we take M equal to the present angular momentum of the Sun, M must be reduced to about one-sixtieth † of its former value, and $k^2/{r_0}^2$ is reduced to the same fraction of the values just calculated.

10. For an incompressible mass rotating so that $\omega^2/2\pi\gamma\bar{\rho}$ is equal to $\cdot 18712$, the value of k^2 is $\cdot 5731 {r_0}^2$, so that it is clear that if our Sun ever rotated with sufficient angular momentum to throw off a satellite, there must have been very extreme concentration of mass near the centre.‡ Practically all the mass must have been

* *C.R.*, lii. (1861), p. 481.

† This is approximately the ratio 1 : 61 . 163 . . . given by See (*Ast. Nach.*, 4053).

‡ See Poincaré, *Les Hypothèses Cosmogoniques*, p. 20.

concentrated within a sphere of radius comparable with k, and so only a small fraction of r_0; and for such a mass the potential at regions near the edge may, with almost perfect accuracy, be taken to be

$$V_M = \frac{M}{R}.$$

The configurations of equilibrium of such a mass rotating freely under no tidal forces have been studied by Roche.* The principal series of configurations consists of figures which are at first spheroids, but give place finally when $\omega^2/2\pi\gamma\bar{\rho} = \cdot 36$ to a lens-shaped figure with a sharp edge. At this point a break-up occurs through $\partial p/\partial n$ vanishing over the edge. Up to this point the configurations are all stable, and there is no point of bifurcation.

It follows that if our system has ever broken up through rotation, the mechanism must have been of what we have called the second kind, matter being thrown off round the equator of the nebula, and the value of ω^2 must have been given by

$$\frac{\omega^2}{2\pi\gamma\bar{\rho}} = \cdot 36 \quad . \quad . \quad . \quad . \quad (12)$$

With this definite knowledge of the value of ω^2 we can correct the calculations of § 9, and the results stand as in the following table :—

Values of k^2/r_0^2 *when* $\omega^2/2\pi\gamma\bar{\rho} = \cdot 36$.

Assumed Value for r_0.	M=	Total Moment of Momentum of Solar System.	Present Angular Momentum of Sun.
Radius of present Sun . . .		·072	·0012
,, orbit of Earth . .		·005	·00008
,, ,, Neptune . .		·00090	·000015

11. If a break-up of the kind we are now considering occurred, we may suppose that when it happened the density of the Sun continually increased as we passed inwards. No other arrangement can be permanently stable, so that exterior matter which has cooled until it has become denser than interior matter must after a short interval change place with the latter.

Let the ejected matter be supposed to have come from near the surface of the Sun, and to have been of density $\beta\bar{\rho}$ before ejection; then the vast majority of matter in the Sun may be supposed to have been of density greater than $\beta\bar{\rho}$. The value of Mk^2 is

$$Mk^2 = \iiint (x^2 + y^2)\,\rho\, dx\, dy\, dz,$$

and this will be greater than

$$\beta \iiint (x^2 + y^2)\,\bar{\rho}\, dx\, dy\, dz,$$

* Poincaré, *loc. cit.*, chap. iii.

or $\beta M k_0^2$, where k_0 is the radius of gyration of a homogeneous mass bounded by the lens-shaped figure. The value of k_0^2 is found to be about $\frac{1}{2}r_0^2$,* so that we necessarily have

$$\frac{1}{2}\beta < \frac{k^2}{r_0^2} \quad . \quad . \quad . \quad . \quad (13)$$

It now appears that the values calculated in the table above provide upper limits to the values of $\frac{1}{2}\beta$. Thus if the Sun ejected matter by rotation, the density of this matter before ejection must have been less than a certain calculable density.

For instance, imagine the Sun ejecting matter when of a mean radius equal to that of Neptune's orbit, its angular momentum being equal to that of the whole present solar system. The value of $\bar{\rho}$ is $5{\cdot}5 \times 10^{-12}$ and $\frac{1}{2}\beta < {\cdot}00090$, so that $\beta\bar{\rho}$ must have been less than 10^{-14}. If the angular momentum was only that of the present Sun, $\beta\bar{\rho}$ must have been less than $1{\cdot}8 \times 10^{-16}$.

Conditions for Condensations to Occur.

12. Matter of very low density, ejected slowly from a rotating mass, is not likely to condense at all; it will scatter into space or fall back into the Sun. For a rotational theory of planetary evolution to be tenable it must show that the ejected matter would be of sufficient density and amount to condense into planets.

Poincaré,† following Roche, assumes that the ejected matter, if it condenses at all, will condense into a ring rotating with the same velocity ω as the primary. He recognises that a condition for condensation to occur at all is that the density shall be above a certain limit, and proves that the ring can only exist without breaking up if its mean density ρ' is greater than $\omega^2/2\pi\gamma$: "Nous avons ainsi une limite inférieure de la densité de l'anneau, et *à fortiori* de la nébuleuse." But the value of $\omega^2/2\pi\gamma$ is known to be ${\cdot}36\bar{\rho}$, and so this lower limit is enormously greater than the upper limit allowed by equation (13). Our Sun can only have ejected matter by rotation if its density were less than, say, ${\cdot}002\bar{\rho}$, while ejected matter can only have formed into planets if its density were greater than ${\cdot}36\bar{\rho}$. The two limits occur in the wrong order, and we conclude that, subject to the assumptions on which we have been working, a rotational theory of planetary formation is untenable as regards our solar system.

The argument can perhaps be seen more forcibly by being reversed. Assume that our Sun broke up under rotation and that condensation occurred. Then $\omega^2/2\pi\gamma$ must have been equal to ${\cdot}36\bar{\rho}$, and must have been less than ρ'. Thus β must have been greater than $\rho'/\bar{\rho}$, and so *à fortiori* greater than ${\cdot}36$. Hence k^2/r_0^2 must have been greater than $\frac{1}{2}\beta$, and so *à fortiori* greater than ${\cdot}18$. Hence from equation (8), r_0 must have been less than ${\cdot}16$ times

* The exact value is ${\cdot}52313r_0^2$.

† *Les Hypothèses Cosmogoniques*, p. 23.

the present radius of the Sun, and the mean density of the primæval nebula must have been greater than 244 times that of the present Sun, or greater than 30 times the density of lead. This assumes the whole present angular momentum of the system to have been concentrated in the central nebula when the last satellite was born; if we suppose the angular momentum to have been double that of the present Sun, the requisite density is found to be many million times that of lead.

13. This conclusion rests on two assumptions. First, there is the assumption, apparently regarded as obviously true by Poincaré, that the ejected matter will expand on ejection; second, there is the assumption that if the matter condenses it will form a ring rotating in equilibrium.

As regards the first assumption, it is natural to suppose that to a first approximation the gas changes adiabatically on ejection, and so necessarily expands. There may be a certain loss of energy by radiation, but seeing that the ejected matter is only supposed to move from just inside the Sun to just outside the Sun, it seems unthinkable that there should be a contraction sufficient to bridge over the great gap between the two limits of density. In any case we shall find (§ 14) that further arguments which do not involve this assumption also show the rotational hypothesis to be untenable.

The second assumption is of a severely restrictive nature, for it implies perfect symmetry about the axis of rotation, both before and after ejection. Every star or mass of matter in the neighbourhood of the rotating body exerts tidal forces which disturb this symmetry. In general the distance of even the nearest stars will be great compared with the dimensions of the rotating mass. The tidal forces may be expanded in a series of harmonics; the terms of first degree will as usual provide the acceleration of the rotating mass in space, and of the remaining terms, those of second degree will adequately represent the tidal potential. Equation (2) now assumes the form

$$\Omega = \frac{M}{R} + \tfrac{1}{2}\omega^2(x^2+y^2) + p_{11}x^2 + p_{12}xy + \ \ldots$$

and on drawing the surfaces $\Omega = \text{cons.}$, which are possible boundaries for the mass, it becomes clear that $\partial\Omega/\partial n$ (or $\partial p/\partial n$) will not vanish simultaneously round a ring, but will first vanish at two points at opposite ends of a diameter. Thus small tidal forces such as occur in nature will localise the area over which matter is ejected, and the ejected matter will form two jets streaming out from antipodal points.

As being applicable to the ejection of matter in this way, it is easy to obtain the equation (a generalisation of equation (4))

$$\frac{d^2W}{dt^2} = \int\int \frac{1}{\rho}\frac{\partial p}{\partial n} dS + 2\omega^2 W - 4\pi M \quad . \quad . \quad (14)$$

where M is a detached mass of ejected matter enclosed in a volume W,

and the surface integral extends over the boundary of this volume. In obtaining this equation, the ejected matter is supposed to move in any way whatever, subject to the condition that velocities of ejection are small compared to the velocity of rotation (ωr), and also to the condition that the vorticity remains everywhere equal to $(0, 0, 2\omega)$, the vorticity of the movement of uniform rotation before ejection.*

In this equation the first term on the right represents the effect of gas-pressure on changes in W; since p must necessarily increase on passing inwards, it appears that gas-pressure tends to increase $d\mathrm{W}/dt$. Similarly the next two terms represent the expanding effect of rotation and the condensing effect of gravitation.

Viscosity is not taken into account in arriving at equation (14). To include the most general system of stresses possible, we must replace the first term on the right by

$$\iint -\frac{1}{\rho}\left\{l\left(\frac{\partial p_{xx}}{\partial x}+\frac{\partial p_{xy}}{\partial y}+\frac{\partial p_{xz}}{\partial z}\right)+m(\ \ldots\ )+n(\ \ldots\ )\right\}d\mathrm{S},$$

where l, m, n are the direction-cosines of the normal; and this must again be positive, each element representing the outward thrust on a certain surface layer of gas.

If the matter expands at first on ejection, $d\mathrm{W}/dt$ will begin by being positive. Equation (14) shows that unless $\mathrm{M}>\omega^2\mathrm{W}/2\pi$, the value of $d\mathrm{W}/dt$ will continually increase, so that W will increase beyond limit, and the matter scatters freely into space. Thus we are brought back to the same limit $\rho'>\omega^2/2\pi$ which we used in § 12, and the conclusions of that section are re-established.

14. The foregoing considerations have been concerned only with the possibility of rotation preventing condensation from occurring. But we have seen that gas-pressure is also an agency opposing condensation. Let us now consider the effect of gas-pressure upon the possibility of a stream of ejected matter condensing into (i) a long filament, and (ii) a series of detached masses.

It is readily shown that a long cylinder or filament of gas in equilibrium under its own gravitation must necessarily have a mass $2\kappa/\gamma$ per unit length, when γ is the gravitation constant and κ is the ratio of pressure to density. This formula assumes the

* Poincaré (*Hypothèses Cosmogoniques*, p. 31) has studied the possibility of a mass persisting permanently without expansion when the vorticity is not uniform. He finds that ω must be a function of $\tilde{\omega}$ (where $\tilde{\omega}^2=x^2+y^2$), and a necessary condition for permanence is that

$$2\pi\rho'-\left(\omega^2+\tilde{\omega}\omega\frac{\partial\omega}{\partial\tilde{\omega}}\right)>0 \qquad . \quad . \quad . \quad . \quad \text{(i)}$$

so that ρ' can only be very small if $\tilde{\omega}\omega$ decreases as $\tilde{\omega}$ increases. Such a condition as this, even if it could ever be established, could only be permanently satisfied in a perfectly non-viscous medium. Viscosity must always tend to equalise values of ω, so that if the inequality (i) is satisfied at one instant, it cannot be so permanently unless $\rho'>\omega^2/2\pi$. So that if ρ' is very small, the mass cannot remain in a condensed state. *Cf.* Poincaré, *l.c.*, p. 49.

matter to be at uniform temperature and away from the influence of tidal forces, but temperature variations and tidal forces will not alter the order of magnitude of the result. In terms of the mean-square molecular velocity C, the mass per unit length becomes

$$\frac{2}{3}\frac{C^2}{\gamma} \quad . \quad . \quad . \quad . \quad (15)$$

As we should expect, the mass per unit length necessary for condensation is greater for high molecular velocities than for low. Gravitation, represented by γ in formula (15), tends to produce condensation, while energy of molecular motion, represented by C^2, tends to resist it. The ratio of these quantities determines whether condensation can occur or not.

A lower limit to the velocity of projection will be fixed by the velocity of effusion into a vacuum, and this is known to be * $C/\sqrt{(6\pi)}$. Thus for a column of gas to condense as it comes out, its rate of emission must be greater than

$$\cdot 153\, C^3/\gamma \text{ gms. per second} \quad . \quad . \quad . \quad (16)$$

Calculations relative to the formation of globular or other detached masses lead by a different path to precisely similar results. It is known that a body of gas can only condense into a permanent globular mass when both its mass and radius are infinite. There are, however, possibilities of large bodies of gas forming into almost permanent masses, the determination of the rate at which these dissolve being similar to that of the dissipation of planetary atmospheres in the kinetic theory of gases.

A molecule moving with velocity c will escape from the surface of a globular mass of mass M and radius R if

$$\tfrac{1}{2}c^2 > \gamma M/R,$$

so that very rapid dissolution can only be avoided if

$$\pi\bar{\rho}R^2 > \frac{3C^2}{8\gamma}$$

The term on the left is the mass of the globe per unit length at its equator; that on the left only differs by a factor $\frac{9}{16}$ from expression (15). Hence the discussion can proceed as before, the factor $\frac{9}{16}$ naturally not materially influencing the result.

From expression (16) it is clear that the rate of emission necessary for condensation will be least when C is least. It is again a question of balancing molecular velocity against gravitation. Taking $C = 3 \times 10^4$, the velocity appropriate for hydrogen at 7° abs., or for oxygen at −156° C., or iron at 120° C., we find that formula (16) demands an emission of matter at a rate greater than 6×10^{19} gms. a second, so that a mass equal to that of the Earth must be ejected in less than three years. The more likely value, $C = 6 \times 10^4$,

* Jeans, *Dynamical Theory of Gases*, p. 133 (2nd edition).

would require the Earth's mass to be emitted in five months. These rates clearly indicate something very different from the slow emission of matter contemplated by the rotational theory. It appears that matter emitted in a series of violent explosions could condense, but the main agency in such a process would be explosion and not rotation. To get condensation from a rotational theory the molecular velocity would have to be far below 3×10^4, and so far below the velocity at the boiling point of most substances.

The only way in which we can get a molecular velocity low enough for condensation to take place is by replacing our molecules by meteorites or particles of dust. But when this is done, and the matter is supposed to have a density comparable with that calculated in § 11, the free path becomes so large that the medium no longer possesses the physical properties of a gas. The particles or meteorites describe orbits freely without mutual interference. The mathematical analysis appropriate for the discussion of this case is identical with that for the motion of a cluster of stars, and there is no possibility of such a cluster breaking into detached masses by rotation.*

15. A nebula rotating at the critical speed might emit matter at a sufficient rate for condensation to occur if the nebula were on a much greater scale than our Sun. If the nebula were acted on by slight tidal forces, we might get matter emitted at two antipodal points and condensing into two long filaments as suggested in § 13. These filaments, when of sufficient length, would be longitudinally unstable, and would condense into nuclei at an average distance apart comparable with †

$$\sqrt{\frac{1}{\pi\gamma\rho}\frac{dp}{d\rho}} = \frac{C}{\sqrt{3\pi\gamma\rho}}.$$

Multiplying this by expression (15), we find as the mass condensing about each nucleus ‡

$$\frac{2C^3}{3\gamma\sqrt{3\pi\gamma\rho}} = \cdot 21 C^3 \gamma^{-\frac{3}{2}} \rho^{-\frac{1}{2}} \quad . \quad . \quad . \quad (17)$$

Taking $p = 10^{-17}$ and $C = 6 \times 10^4$ we obtain a mass of about 10^{33} grammes, and therefore comparable with our Sun. To obtain a mass as small as Uranus or Neptune, ρ would have to be about 10^{-9}, about 200 times greater than the mean density of the mass of the Sun spread through a sphere of radius that of Neptune's orbit, or again about 200,000 times the maximum density allowed by our calculations of § 11, on supposing the nebula to extend to Neptune's orbit.

* *M.N.*, lxxvi. (1915), p. 78.

† See the paper already referred to "On the Motion of Tidally distorted Masses."

‡ This formula, except for the numerical coefficient, can be obtained from a consideration of physical dimensions only, so that it may be taken to express the mass formed by any process whatever of gravitational condensation.

Conclusion.

16. To sum up, it has been found that the rotational theory fails to explain the genesis of the solar system in three distinct ways:—

(i) The present momentum of the system fixes a limit to the density near the edge of the primæval nebula, and this density is so low that matter thrown off from the edge could not condense into planets.

(ii) The only form of rotational theory which is applicable to our system postulates matter being thrown off continuously and slowly, and the matter would be ejected so slowly that it could not condense.

(iii) Even if these objections were not valid, and matter had condensed, the masses formed would have been much greater than the planets of our system.

The discussion has necessarily been dependent upon numerical estimates, but we have been able to argue in terms of upper and lower limits. Some of these limits have, however, been very wide, and each time we have used the limit most favourable to the rotational hypotheses, we have in effect weighted the beam of the balance so as to favour this hypothesis unduly. Weighed in this balance, the hypothesis has been found wanting. It is impossible to conjecture whether or not in the course of time new factors or new physical agencies may appear of which our argument has taken no account. If such agencies appear, the balance may conceivably again swing in the other direction. In particular, it has been assumed throughout that the angular momentum of the primitive nebula cannot have been greater than the present total angular momentum of the whole solar system. Thus if any sufficiently effective means of leakage of angular momentum comes to light in the future, the whole argument will require reconsideration. But, on the present evidence, we can say with some confidence that the rotational theory cannot explain the origin of the solar system.

In general, it has become clear that detached masses can be formed by rotation in two cases only:—

(i) When the angular momentum is much greater than that in our system, a condition which appears to lead to the binary-star formation.

(ii) When the mass is much greater than that in our system. This condition appears likely to lead to the spiral nebula formation.

Note on the Action of Viscosity in Gaseous and Nebular Masses.
By J. H. Jeans, M.A., F.R.S.

1. The general equations of motion of a viscous mass are

$$\rho\frac{Du}{Dt}=\rho X-\frac{\partial p}{\partial x}+\tfrac{1}{3}\kappa\frac{\partial\theta}{\partial x}+\kappa\nabla^2 u, \text{ etc.} \quad . \quad . \quad (1)$$

where κ is the coefficient of viscosity, p is the mean pressure, and θ the expansion, these being defined by

$$p=\tfrac{1}{3}(p_{xx}+p_{yy}+p_{zz}),$$
$$\theta=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z},$$

and therefore being of course invariant for changes of axes. For a gas p is roughly proportional to ρ, so that the terms in equation (1) divide themselves into terms multiplied by κ and terms multiplied by ρ. The well-known result follows,* that the effect of viscosity in modifying a given motion is proportional to κ/ρ, the quantity which Maxwell calls the "kinematic" coefficient of viscosity. For a gas at low density the value of κ is independent of ρ, so that κ/ρ will vary inversely as ρ, and may become enormously large at low densities. It follows that viscosity may quite conceivably preponderate over gravitation in determining the motion of a nebulous mass of gas, just as, for instance, it does in determining the motion of particles of dust in the atmosphere.

We shall be concerned mainly with the internal motions in a rotating mass of gas, so that it will be convenient to transform equation (1) to rotating axes, which we shall suppose to rotate with a uniform velocity ω about the axis of z. The values of the velocities and accelerations are

$$u=\dot{x}-y\omega,\ v=\dot{y}+x\omega,\ w=z,$$
$$\frac{Du}{Dt}=\ddot{x}-2\dot{y}\omega-\omega^2 x,\ \frac{Dv}{Dt}=\ddot{y}+2\omega\dot{x}-\omega^2 y,\ \frac{Dw}{Dt}=\ddot{z}.$$

Let us assume the velocities $\dot{x}$, $\dot{y}$, $\dot{z}$ relative to the moving axes to be small, and let us now denote these by u, v, w, then it is found that equations (1) transform into

$$\rho\frac{du}{dt}=\rho(X+\omega^2 x)-\frac{\partial p}{\partial x}+\tfrac{1}{3}\kappa\frac{\partial\theta}{\partial x}+\kappa\nabla^2 u, \text{ etc.} \quad . \quad (2)$$

Let the vorticities of the motion relative to the moving axes be denoted by ξ, η, ζ, so that

$$\xi=\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}, \text{ etc.}$$

* *Cf.* Lamb, *Hydrodynamics* (4th edition), p. 470.

Assuming that κ is constant throughout the mass, we readily obtain from equations (2), three equations of the type

$$\frac{\partial\rho}{\partial y}\left(\frac{du}{dt} - X - \omega^2 x\right) - \frac{\partial\rho}{\partial x}\left(\frac{dv}{dt} - Y - \omega^2 y\right) = \rho\frac{d\zeta}{dt} - \kappa\nabla^2\zeta \quad . \quad (3)$$

Multiplying these three equations by $\partial\rho/\partial x$, $\partial\rho/\partial y$, $\partial\rho/\partial z$, and adding, we obtain

$$\sum\frac{\partial\rho}{\partial z}\left(\rho\frac{d\zeta}{dt} - \kappa\nabla^2\zeta\right) = 0 \quad . \quad . \quad . \quad (4)$$

2. The simplest case arises when the whole vorticity is everywhere about the axis of z. We then have $\xi = \eta = 0$, and equation (4) becomes

$$\frac{d\zeta}{dt} = \frac{\kappa}{\rho}\nabla^2\zeta \quad . \quad . \quad . \quad . \quad (5)$$

which is the well-known Fourier equation.

Thus the vorticity ζ is propagated according to the same law as governs the conduction of heat or diffusion. The similarity extends beyond the general nature of the law, for the coefficient κ/ρ is very nearly equal to the corresponding coefficient $\mathfrak{D}$ in the equation of diffusion of gases,* and also to the coefficient θ/c in the equation of conduction of heat in gases.†

For instance, if the outer layers of a nebula shrink through cooling, ζ will be increased at the surface, and this excess value of ζ will be propagated inwards until the value of ζ is uniform, just in the same way as an excess of temperature at the boundary would be propagated inwards until the temperature was uniform. A continual shrinkage through cooling at the surface will be analogous to a continual increase of temperature at the surface, and the vorticity ζ will acquire a steady gradient exactly similar to a temperature gradient.

Unfortunately this special case of $\xi = \eta = 0$ is of only limited interest, as we shall now see. For if $\xi = \eta = 0$, then equation (5) must be satisfied, and hence the right-hand members of all the three equations (3) disappear. It follows that we must have

$$\frac{du}{dt} - X - \omega^2 x = q\frac{\partial\rho}{\partial x} \quad . \quad . \quad . \quad (6)$$

and two similar equations, from which we readily deduce

$$\frac{d\xi}{dt} = \frac{\partial\rho}{\partial z}\frac{\partial q}{\partial y} - \frac{\partial\rho}{\partial y}\frac{\partial q}{\partial z} \quad . \quad . \quad . \quad . \quad (7)$$

and two symmetrical equations. The vanishing of $d\xi/dt$ and of $d\eta/dt$ is now seen to require that $d\zeta/dt$ shall also vanish, except in very special cases, so that in general the motion reduces to a

* Jeans, *Dynamical Theory of Gases* (2nd edition), p. 326.
† *Ibid.*, p. 307.

steady motion in which propagation of vorticity does not occur at all.

We are accordingly compelled to fall back on the general equation (4), of which no exact interpretation presents itself. It will, however, be clear that in a general way vorticity is propagated at a rapidity comparable with that expressed by equation (5), and therefore comparable with the propagation by diffusion or conduction of heat.

In the special case of ρ being constant everywhere, it is of course obvious from equation (3) that each component of vorticity satisfies equation (5) exactly.

3. It is obvious without discussion that the general motion determined by equations (2) and (3) must represent a compromise between the motion when there is no viscosity ($\kappa = 0$) and the motion when viscosity is predominant ($\kappa = \infty$). Let us examine the abstract limiting case of $\kappa = \infty$.

Equations (3) assume the form $\nabla^2\xi = 0$, etc., and it is clear that the case of main practical interest occurs when ξ, η, ζ are all constants. By a new and suitable choice of the axes and amount of their rotation ω, these values of ξ, η, ζ can all be made to vanish, and we are left with an irrotational motion, relative to these moving axes, of components u, v, w given by

$$u = -\frac{\partial\phi}{\partial x}, \quad v = -\frac{\partial\phi}{\partial y}, \quad w = -\frac{\partial\phi}{\partial z} \qquad . \quad . \quad (8)$$

When $\kappa = \infty$ equations (2) assume the form

$$\tfrac{1}{3}\frac{\partial\theta}{\partial x} + \nabla^2 u = 0, \text{ etc.},$$

and in virtue of relations (8), these are equivalent to

$$\frac{\partial}{\partial x}\nabla^2\phi = 0, \text{ etc.}$$

Thus $\nabla^2\phi$ must be a constant, say $-A$, and ϕ must be of the form

$$\phi = -\tfrac{1}{6}A(x^2 + y^2 + z^2) + \text{a spherical harmonic.}$$

We have

$$\frac{1}{\rho}\frac{D\rho}{Dt} = -\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = -A,$$

so that each element of gas expands at a uniform rate A, undergoing at the same time a change of shape which depends on the spherical harmonic part of ϕ.

4. According to the suggestion of the preceding paper, a spiral nebula may be regarded as made up of a lens-shaped nucleus, surrounded by an atmosphere of gas emitted from the periphery of the lens, the emission being almost entirely localised at two antipodal points, and the atmosphere being accordingly almost confined to the

equatorial plane of the mass. At most only the motion of this outer atmosphere will approximate to that just considered.

The value of the spherical harmonic in ϕ appropriate to such a case is readily seen to be

$$-\tfrac{1}{12}\mathrm{A}(x^2+y^2-2z^2)$$

so that

$$\phi = -\tfrac{1}{4}\mathrm{A}(x^2+y^2),$$

and the velocities are given by

$$\dot{x} = \tfrac{1}{2}\mathrm{A}x, \quad \dot{y} = \tfrac{1}{2}\mathrm{A}y, \quad \dot{z} = 0.$$

If the rate of emission of matter is uniform, A will be independent of the time, and the equations of the orbit described by any element of the emitted gas will be

$$x = x_0 e^{\frac{1}{2}\mathrm{A}t}, \quad y = y_0 e^{\frac{1}{2}\mathrm{A}t}.$$

Thus, relative to the moving axis, each element describes a straight line with a velocity which increases exponentially with the time. The orbit in space will be

$$r = r_0 e^{\frac{1}{2}\mathrm{A}\theta/\omega},$$

an equiangular spiral. If the matter is emitted from two antipodal points, the main mass of the ejected matter will of course describe two equiangular spirals starting from these two points, the matter expanding as it proceeds so that the boundaries of the jet are themselves equiangular spirals.

5. The motion just considered is only the limiting case in which viscosity preponderates entirely over gravitation, etc. Actual motion must be intermediate between this and the motion obtained by neglecting viscosity altogether and supposing the orbits determined by gravitation and gas-pressure. But inasmuch as the arms of many spiral nebulæ approximate much more closely to equiangular spirals * than to gravitational orbits, the question arises as to what numerical conditions must be satisfied in order that the purely viscous motion may give a good approximation to the truth.

We have seen that the vorticities are propagated approximately in accordance with equation (5), so that the time necessary for the vorticity approximately to equalise itself over a distance r must be comparable with

$$\frac{\rho r^2}{\kappa} \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Remembering that κ is independent of the density in a gas, it might be thought that propagation could be made as rapid as we please by taking ρ small enough. But this does not prove to be the case. The value of κ is approximately $\frac{1}{3}\rho cl$, where c, l are molecular velocity and free-path, so that formula (9) reduces to $3r^2/cl$. This formula is, however, only valid when l is small

* v. Pahlen, *Ast. Nach.*, 4503.

compared with r, and for values of l greater than r the effect of viscosity falls off with extreme rapidity. Thus the time of equalisation must always be greater than r/c, and there can be no approach to equalisation unless the radial velocity of the arms is much smaller than the molecular velocity, a condition which is not likely to be satisfied in a real nebula.

The result just obtained is only an illustration of the obvious principle that the angular velocity in the arms can only tend to approximate to that of the central mass through some agency which moves with a speed greater than the radial velocity of the arms. Molecular motion, then, is almost certainly inadequate; the motion of free electrons might possibly provide the needed agency, as also might electric forces (acting, for instance, through electric charges in the medium).

Perhaps, however, it is wiser to leave discussion of the general question until something definite is known, or something reasonable has been conjectured, as to the physical constitution of nebular masses. For the present it may be recorded that it has been found that viscosity (possibly of a somewhat generalised kind) can be highly effective in influencing the paths of matter in a nebula, and that the paths, if determined by viscosity alone, would be equiangular spirals.

Greenwich 1910 *Catalogue of Stars, Zone* $+24^\circ$ 0′ *to* $+32^\circ$ 0′: *List of Stars the P.M. of which is* 2″ *or over per century; and Analysis of Numbers and Percentages of Proper Motions.* By W. G. Thackeray.

(*Communicated by the Astronomer Royal.*)

The Greenwich 1910 Catalogue is a complete survey of the zone of the sky covering the region $+24^\circ$ 0′ to $+32^\circ$ 0′ of all stars down to B.D. magnitude 9·0.

The proper motions have been determined of over 12,000 stars. About 7500 of these were observed by Bessel about 1825, and 5500 by Lalande about 1800, and all of them were observed in the Astronomische Gesellschaft catalogues of Leiden, Cambridge, and Berlin B. These catalogues have been all reduced to the basis of Boss's system. Full particulars of the systematic corrections applied will be given in the introduction to the Catalogue when the exigencies of war allow of the printing. The following table gives the stars of largest proper motion. It requires no explanation beyond the * in column 1, which denotes that the star is in Boss's Preliminary General Catalogue, and the proper motion is taken from that source. In the column A.G.C. No. the numbers without a letter refer to the Cambridge Catalogue, the letters B and L to Berlin and Leiden Catalogues respectively.

TABLE I.

List of Stars the P.M. of which is 20″ or more per Century.

B.D. No.		A.G.C. No.	Mag.	Approximate. R.A. 1910. h m s	Dec. ° ′	P.M. R.A. Time (s·0001).	R.A. Arc. (″·001).	Dec. ″·001.
28° 4704*		3	7·5	0 1 56	28 32	+288	+380	− 176
28 4*		19	2·0	3 44	28 36	+106	+139	− 161
24 4	B	15	7·8	4 36	24 47	+143	+195	− 142
29 14		37	8·5	6 42	29 57	+163	+212	+ 35
26 8		47	8·5	7 23	26 36	+222	+298	− 60
29 31		95	8·7	11 29	30 2	+142	+172	+ 49
30 34		120	8·6	13 38	30 28	+102	+133	− 169
25 77		328	9·0	30 0	25 34	+285	+388	− 57
29 105		344	8·8	31 14	29 30	+125	+164	− 419
28 103*		370	4·1	33 48	28 49	−173	−227	− 248
30 89		372	9·0	34 1	30 32	−170	−218	− 70
29 141		486	7·5	45 0	29 58	+169	+220	− 31
30 125	L	288	8·0	47 46	30 58	+175	+225	− 46
29 158		574	8·2	53 7	29 50	+162	+212	− 120
30 206	L	494	8·8	1 16 23	30 52	+380	+487	− 95
27 225		800	7·7	20 11	28 6	+205	+271	− 165
31 247	L	521	8·7	20 27	31 31	−261	−334	− 102
30 233	L	546	8·1	24 47	30 33	+163	+209	0
28 249		849	7·3	26 16	28 57	+149	+195	− 78
28 253		867	8·8	28 17	28 48	+189	+248	− 45
24 239	B	490	7·4	32 27	24 43	−155	−211	− 238
30 252		904	8·0	32 58	30 20	+194	+252	+ 5
28 271		906	8·8	33 17	29 7	+316	+414	− 238
27 262		918	7·8	34 30	27 39	+371	+490	+ 168
28 312*		1022	4·3	47 57	29 8	+ 13	+ 17	− 232
30 308	L	733	8·9	52 32	31 9	+171	+219	− 76
27 335		1146	9·0	2 3 12	27 53	+129	+170	− 304
29 366		1158	8·7	5 10	29 23	+217	+284	− 261
26 373		1200	8·0	10 15	26 56	+234	+313	− 136
30 421	L	990	7·3	33 10	30 26	−368	−478	− 404
25 449		1436	8·0	41 0	25 16	+177	+240	− 183
26 465		1454	8·1	42 50	26 42	+218	+291	− 119
30 448	L	1049	7·2	43 18	30 45	+163	+209	− 168
26 484		1512	7·6	50 19	26 31	+192	+257	− 162
29 503		1523	9·0	52 22	29 18	+157	+205	− 54

TABLE I.—*List of Stars the P.M. of which is 20″ or more per Century*—contd.

B.D. No.		A.G.C. No.	Mag.	Approximate. R.A. 1910. h m s	Approximate. Dec. 1910. ° ′	P.M. R.A. Time (s·0001).	P.M. R.A. Arc (″·001).	P.M. Dec. ″·001.
26°	503*	1553	7·1	2 57 6	26 15	+179	+241	− 168
25	495	1581	8·1	3 3 8	26 0	−131	−176	− 836
25	580	1758	8·0	31 46	25 42	+185	+250	− 259
29	597	1805	8·6	37 28	29 54	+142	+185	− 157
28	593	1858	8·5	43 54	28 22	+ 48	+ 63	− 225
31	660	L 1451	8·8	46 30	31 45	+ 87	+111	− 354
30	609	L 1516	8·2	56 45	31 5	− 76	− 98	− 196
28	614	1967	8·1	58 14	28 57	+213	+279	− 122
29	678	2025	7·0	4 9 31	29 41	+ 85	+111	− 170
30	653	L 1646	8·7	14 24	30 56	+ 88	+113	− 181
24	659	B 1436	8·9	22 24	24 14	+263	+360	+ 102
26	730	2118	8·6	31 23	26 57	+158	+211	− 142
27	688	2150	8·0	37 59	27 31	+ 55	+ 73	− 244
28	726	2254	8·8	54 51	29 4	+209	+274	− 134
24	726	B 1602	8·3	56 7	24 31	+ 76	+103	− 263
27	733	2321	8·9	5 4 22	27 32	+145	+192	− 49
27	734	2322	7·3	4 25	27 27	+153	+205	− 104
24	772	2330	8·3	5 13	25 2	− 26	− 35	− 210
27	881	2718	8·9	44 7	27 6	+ 68	+ 90	− 256
31	1159	L 2392	9·0	54 30	31 26	+129	+166	− 286
26	1067	2935	8·9	6 0 20	26 34	−119	−159	− 357
29	1154*	3064	4·4	9 30	29 31	− 49	− 64	− 266
27	1124	3265	8·7	23 30	27 5	−151	−202	− 429
24	1357	B 2526	8·5	35 45	24 3	+139	+190	− 280
25	1460	3536	7·2	45 27	25 52	− 12	− 15	− 210
30	1359	L 2875	8·1	49 42	30 17	+191	+248	− 245
29	1441*	3695	6·3	57 47	29 29	+121	+158	− 823
30	1423	3753	8·6	7 2 22	29 59	−111	−144	− 319
25	1594	3774	7·3	4 4	25 53	− 81	−109	− 203
25	1613	3831	8·6	8 26	25 10	−294	−399	− 104
27	1356	3880	8·1	12 52	27 25	+119	+159	− 139
32	1562*	L 3150	4·8	23 19	31 57	+117	+141	− 183
29	1539	3991	8·8	24 43	29 34	− 1	− 1	− 206
25	1709	4051	8·0	29 21	25 10	+ 96	+130	− 352
29	1590*	4132	5·0	37 41	29 6	+ 53	+ 69	− 237
28	1463*	4150	1·3	39 49	28 14	−471	−622	− 58

TABLE I.—*List of Stars the P.M. of which is 20″ or more per Century*—contd.

B.D. No.		A.G.C. No.	Mag.	Approximate. R.A. 1910. h m s	Approximate. Dec. 1910. ° ′	P.M. R.A. Time (s·0001).	P.M. R.A. Arc (″·001).	P.M. Dec. ″·001.
31° 1684	L	3313	8·2	7 47 48	30 53	+563	+721	−1848
29 1664		4285	7·5	54 58	29 29	−102	−134	−1179
25 1865*		4397	6·0	8 5 2	25 46	− 51	− 58	− 354
31 1781	L	3472	8·3	12 37	30 54	−227	−291	− 814
27 1589*		4482	5·7	14 36	27 30	− 9	− 12	− 388
24 1956		4603	9·0	31 30	24 47	−159	−216	+ 44
26 1816		4618	7·5	32 47	26 22	− 85	−115	− 186
30 1739		4619	9·0	32 52	29 57	−183	−238	− 105
25 1981		4674	8·5	39 6	25 8	− 96	−130	− 337
28 1660*		4722	6·0	47 14	28 40	−365	−479	− 245
26 1865		4732	7·0	49 35	26 33	+ 54	+ 72	− 437
27 1715*		4838	6·0	9 3 30	27 0	− 87	−116	− 387
25 2059		4860	8·9	6 39	25 26	−169	−229	− 104
29 1883		4896	7·0	12 33	28 57	+ 58	+ 75	− 509
31 1978	L	3860	7·5	20 7	30 53	+ 74	+ 95	− 190
27 1775		5001	7·0	27 30	27 24	−108	−144	− 236
26 2019*		5130	4·0	47 39	26 25	−163	−220	− 63
28 1826		5173	7·9	54 9	27 57	−263	−347	− 57
25 2191		5179	7·5	54 58	24 59	−190	−258	− 53
27 1844		5202	8·7	10 0 22	26 56	−164	−219	− 90
29 2000		5216	8·0	2 5	29 41	+ 22	+ 29	− 244
24 2193	B	3953	8·3	6 49	24 12	−291	−398	+ 62
26 2065		5280	7·5	11 53	26 19	+114	+153	− 299
25 2247		5341	8·1	20 2	25 4	+ 15	+ 21	− 203
25 2255		5365	8·9	23 20	25 20	+147	+199	− 130
27 1914		5430	8·5	33 11	27 5	+161	+215	− 88
28 1923		5478	8·5	38 57	27 53	−175	−232	− 80
26 2148		5557	8·9	51 12	26 10	−156	−210	− 94
28 1952		5559	8·4	51 25	28 13	−341	−450	− 145
29 2108		5563	8·8	52 13	29 25	+247	+323	− 169
25 2335*		5620	7·9	11 0 27	25 41	−297	−400	− 82
31 2238*			8·7	6 6	30 56	+452	+579	− 212
31 2240	L	4377	8·5	6 8	30 56	+452	+579	− 212
29 2156		5740	8·7	20 14	29 30	−234	−306	− 155
24 2360		5764	8·5	22 48	24 32	−156	−212	− 111
30 2163	L	4449	6·8	25 18	30 28	− 70	− 91	− 192

TABLE I.—*List of Stars the P.M. of which is* 20″ *or more per Century*—contd.

B.D. No.		A.G.C. No.	Mag.	Approximate. R.A. 1910. h m s	Dec. ° ′	P.M. R.A. Time (s·0001).	R.A. Arc (″·001).	Dec. ″·001.
30° 2164	L	4450	9·0	11 25 21	30 29	− 81	−105	− 223
28 2032		5845	8·8	38 2	27 45	−153	−202	− 57
31 2290	L	4509	8·5	39 59	31 28	− 44	− 56	− 391
29 2253		5983	8·0	12 1 1	28 48	+124	+162	− 121
30 2238	L	4619	8·7	8 8	30 30	−136	−177	− 129
28 2106		6067	6·5	14 59	28 40	−151	−197	− 142
25 2493		6079	7·3	16 33	25 32	−169	−227	+ 158
32 2241	L	4666	8·6	20 2	31 47	−156	−199	+ 16
27 2135		6121	8·2	22 43	27 31	+ 30	+ 40	− 242
30 2310		6194	8·7	34 48	29 44	+112	+146	− 200
31 2401	L	4752	8·9	38 58	31 18	+ 50	+ 64	− 202
24 2493		6241	6·7	42 7	24 39	− 91	−123	− 241
27 2173		6252	8·1	44 11	27 5	−199	−267	− 75
25 2568		6253	6·0	44 24	25 20	−244	−331	− 118
29 2330		6267	8·4	46 59	29 21	−116	−153	− 209
27 2187		6280	7·4	48 40	27 17	−172	−229	− 95.
24 2539	B	4698	8·0	13 3 20	24 29	−222	−304	+ 121
24 2540	B	4699	8·2	3 21	24 50	−208	−282	+ 119
30 2371		6368	8·1	4 20	29 52	+147	+191	− 80
28 2193*		6390	4·1	7 40	28 19	−604	−795	+ 875
28 2205		6416	8·3	12 3	28 13	+ 39	+ 52	− 357
25 2619		6430	8·0	14 35	25 5	−223	−303	+ 44
29 2405		6460	8·6	19 23	29 42	−347	−451	+ 242
28 2232		6500	8·7	26 11	28 7	− 59	− 78	− 188
25 2643		6514	6·0	28 32	24 49	+ 48	+ 64	− 219
31 2540	L	5000	6·3	42 11	31 21	−196	−252	− 90
27 2296		6607	7·0	44 58	27 26	−365	−483	− 81
28 2272		6633	8·5	48 57	28 16	+164	+216	− 412
29 2499		6729	8·9	14 4 30	29 25	−165	−217	− 41
30 2494	L	5127	7·3	10 2	30 38	−307	−393	+ 158
30 2512	L	5169	8·3	18 4	30 3	−515	−670	− 322
24 2733	B	5072	9·0	21 33	24 3	+566	+775	−1099
25 2782		6848	8·0	24 14	24 54	−174	−236	− 141
30 2536*	L	5230	5·0	30 46	30 8	+149	+194	+ 120
24 2757	B	5130	8·7	37 17	24 16	−159	−218	+ 6
27 2411		6948	9·0	39 24	27 8	−216	−289	+ 5
26 2606		6981	8·8	45 4	26 5	− 25	− 34	− 326

TABLE I.—*List of Stars the P.M. of which is 20″ or more per Century*—contd.

B.D. No.		A.G.C. No.	Mag.	Approximate. R.A. 1910. h m s	Approximate. Dec. 1910. ° ′	P.M. R.A. Time (s·0001).	P.M. R.A. Arc (″·001).	P.M. Dec. ″·001.
25° 2873*		7084	4·9	15 3 21	25 13	+138	+187	− 183
26 2677		7155	8·0	15 10	26 1	−401	−541	− 116
31 2724	L	5461	7·3	17 11	31 48	−162	−206	+ 143
30 2653*	L	5478	5·2	19 29	30 36	+101	+129	− 198
28 2503		7420	8·0	54 55	27 59	−575	−760	+ 304
25 3020		7468	7·5	16 0 20	25 29	−390	−528	+ 698
26 2854		7688	8·5	27 52	26 38	− 73	− 97	− 253
25 3097		7703	7·8	29 37	25 53	+168	+226	− 65
31 2873	L	5861	7·2	32 57	31 8	− 5	− 7	− 502
29 2860		7742	8·0	34 44	29 51	− 33	− 43	− 198
31 2884*	L	5889	2·6	37 54	31 45	−365	−464	+ 385
24 3137	B	5874	8·2	17 6 56	24 38	−164	−222	+ 223
29 2979		8118	8·0	14 10	29 19	− 53	− 69	+ 208
31 3027	L	6200	8·0	23 21	31 9	−233	−298	+ 89
26 3023		8207	7·9	23 59	26 52	− 55	− 74	+ 280
29 3029		8236	9·0	25 52	29 28	−164	−215	− 288
27 2888*		8419	3·3	42 56	27 46	−244	−322	− 750
29 3165		8592	7·5	57 11	29 34	− 92	−120	+ 163
26 3151		8618	7·3	58 51	26 20	+277	+373	− 581
30 3113	L	6474	6·7	18 0 28	30 23	− 15	− 20	− 288
29 3213		8758	7·0	11 14	29 11	+ 22	+ 29	− 240
26 3215		8797	9·0	14 11	26 38	+243	+325	+ 99
28 3039		9090	8·2	34 45	28 51	− 49	− 64	− 461
31 3330	L	6797	8·7	37 30	31 28	+ 61	+ 78	− 835
25 3780		9797	8·0	19 14 10	25 12	+142	+193	+ 226
24 3737*	B	7012	6·3	21 42	24 45	−137	−186	− 631
31 3618	L	7391	7·0	25 53	31 26	− 13	− 17	− 425
31 3727	L	7589	8·3	38 59	31 32	− 8	− 10	− 216
24 3849	B	7229	7·2	39 19	24 23	+ 59	+ 81	− 293
27 3516		10423	7·1	44 21	27 38	− 4	− 5	+ 225
27 3523		10444	7·0	45 27	27 31	+ 59	+ 78	− 223
29 3815		10636	8·8	53 12	29 18	− 86	−113	− 184
29 3818		10644	8·9	53 34	29 41	+ 85	+111	− 194
29 3820		10646	8·2	53 36	29 35	+ 88	+114	+ 231
29 3872*		10780	6·2	59 55	29 39	+515	+682	− 528
25 4090		10814	7·9	20 1 8	25 48	+ 22	+ 30	− 378
31 3926	L	7926	9·0	1 13	31 47	+157	+208	− 96

TABLE I.—*List of Stars the P.M. of which is* 20″ *or more per Century*—contd.

B.D. No.	A.G.C. No.	Mag.	Approximate. R.A. 1910.	Approximate. Dec. 1910.	P.M. R.A. Time (s·0001).	P.M. R.A. Arc (″·001).	P.M. Dec. ″·001.
°			h m s	° ′			
29 4135	11623	8·7	20 37 33	29 52	+170	+221	− 34
24 4251	B 7952	8·8	45 13	24 37	−133	−181	− 220
31 4230	L 8505	8·3	45 37	31 57	−140	−178	− 279
25 4422	11941	7·0	53 44	26 3	+187	+252	+ 86
26 4070	12099	8·5	21 1 30	26 55	−142	−190	− 141
26 4156	12483	8·4	21 32	26 50	−111	−150	− 158
24 4411	12550	8·7	25 14	25 3	− 93	−126	− 164
26 4237	12786	7·2	37 2	26 21	+263	+355	− 87
25 4607	12834	8·2	40 5	26 6	+151	+203	+ 60
28 4169*	12835	4·5	40 7	28 20	+214	+282	− 240
28 4210	12951	8·9	47 0	28 45	+165	+217	+ 191
31 4574	L 9192	7·6	50 47	31 54	+159	+203	− 250
25 4648	13020	8·8	52 14	26 1	+155	+209	+ 23
29 4550*	13062	7·4	54 42	29 24	−295	−387	− 378
27 4252	13179	8·0	22 1 48	27 32	+ 39	+ 52	− 209
24 4533*	B 8525	4·0	2 49	24 54	+220	+299	+ 18
28 4312	13269	8·7	7 59	28 37	+176	+231	+ 119
24 4563	B 8578	8·9	11 44	24 29	−149	−204	− 85
30 4774	L 9624	8·7	37 56	31 14	+100	+128	− 197
29 4742	13664	8·1	39 25	29 37	+187	+243	+ 23
29 4753	13685	7·2	41 22	29 59	−192	−250	− 348
25 4808	13695	9·0	41 55	25 29	−157	−213	− 12
27 4480*	13858	2·2–2·7	59 25	27 35	+144	+192	+ 133
26 4580*	13952	6·5	23 7 27	26 21	−144	−194	− 122
26 4586	13967	8·9	8 46	26 24	+132	+178	− 118
29 4890	13996	8·0	11 53	30 11	+270	+351	+ 92
28 4562	14030	8·5	15 28	28 23	+497	+655	− 45
30 4961	L 9973	8·4	25 46	31 13	+140	+179	− 164
30 4982	L 10006	7·5	30 57	30 31	+430	+554	+ 239
30 4983	L 10007	7·9	30 59	30 40	−164	−210	− 269
29 4971	14180	7·7	32 36	30 10	+152	+198	+ 63
27 4588	14195	6·8	34 59	27 45	+231	+306	+ 231
28 4634	14250	8·6	40 32	29 4	+701	+918	+ 22
29 5007	14313	9·0	45 53	29 51	+200	+260	− 3
27 4642	14346	7·0	50 26	28 8	+438	+578	+ 2
27 4663	14402	9·0	56 23	27 23	+173	+231	+ 5
26 4734*	14413	6·9	57 28	26 36	+622	+837	− 986

The following tables of numbers and percentages of proper motion are arranged in order of magnitude and size of proper motion. They practically explain themselves, but perhaps attention may be called again to the fact that the stars with proper motion ranging between 5″ and 10″ per century are the most symmetrically distributed among all magnitudes, and that out of 2805 stars with proper motions over 5″ per century, only 77 of these stars are brighter than $6^{m}\cdot0$, and consequently that there are a very large number of faint stars whose mean parallactic motion is equivalent to those of the brightest stars and whose parallax must be sensible. These stars can only be determined by a rigid examination of all faint stars with large P.M.'s. Roughly, between 20 and 25 per cent. of stars down to the $9^{m}\cdot0$ have proper motions over, and 75 to 80 per cent. under, 5″ per century.

TABLE II.

Proper Motions of Stars arranged according to Magnitude and Size of Proper Motion.

Limits of Magnitude.	0″–5″.	5″–10″.	10″–15″.	15″–20″.	20″–30″.	30″–50″.	50″–.	Total No. of Stars.
			Numbers.					
m m								
0–5·9	88	32	19	9	9	4	4	165
6·0–9	290	87	52	15	8	4	5	461
7·0–9	1141	268	92	18	23	17	10	1569
8·0–5	2473	493	120	41	34	23	8	3192
8·6–9	2956	569	123	39	31	14	4	3736
9·0–	2290	495	85	31	14	5	1	2921
Sums	9238	1944	491	153	119	67	32	12044
			Percentages.					
m m								
0–5·9	54	19	11	5·4	5·4	3·0	1·8	
6·0–9	63	19	11	3·2	1·7	0·9	1·1	
7·0–9	73	17	6	1·2	1·5	1·0	0·7	
8·0–5	77·5	15·5	4	1·3	1·1	0·7	0·2	
8·6–9	79	15	3	1·4	0·8	0·4	0·1	
9·0–	78·5	17	3	1·1	0·5	0·1	0·1	

The following table shows the number of stars of mags. $6^{m}\cdot0$ to $9^{m}\cdot0$ with proper motions between 5″ and 10″ per century compared with the total number of stars for each hour of right ascension. The zone cuts the Milky Way about 6^{h} and 19^{h}.

TABLE III.

Number of Stars of Magnitudes $6^{m}\cdot0$ *to* $9^{m}\cdot0$ *the Proper Motions of which are within the limits* 5″ *to* 10″ *for each hour of R.A.*

Limits of R.A.	$6^{m}\cdot0$–9.	$7^{m}\cdot0$–9.	$8^{m}\cdot0$–5.	$8^{m}\cdot6$–9.	$9^{m}\cdot0$–	Sum.	Total No. of Stars $6^{m}\cdot0$–$9^{m}\cdot0$.	Per-centage.
h h								
0–1	4	15	21	30	22	92	486	19
1–2	3	14	23	17	21	78	440	18
2–3	4	10	19	29	26	88	393	22
3–4	6	14	23	21	20	84	407	21
4–5	3	10	14	21	14	62	332	19
5–6	1	12	16	29	17	75	645	12
6–7	4	13	31	·25	19	92	731	13
7–8	2	8	13	25	19	67	512	13
8–9	4	11	16	16	20	67	419	16
9–10	1	10	14	23	8	56	381	15
10–11	4	11	19	32	15	81	359	23
11–12	6	16	14	15	22	73	301	24
12–13	3	14	19	26	13	75	338	22
13–14	8	7	26	17	21	79	313	25
14–15	3	11	23	30	26	93	339	27
15–16	2	7	29	13	23	74	345	21
16–17	5	16	16	21	11	69	371	19
17–18	3	13	23	16	23	78	530	15
18–19	1	8	28	27	32	96	686	14
19–20	5	14	21	32	30	102	972	10
20–21	4	7	29	26	22	88	849	10
21–22	1	9	16	22	24	72	653	11
22–23	2	9	22	32	30	95	571	17
23–0	8	9	18	24	17	76	507	15
Sums	87	268	493	569	495	1912	11880	
Means	4	11	21	24	21	80		

A Statistical Discussion of the Proper Motions of the Stars in the Greenwich Catalogue for 1910. By Sir F. W. Dyson, F.R.S.

In this number of the *Monthly Notices* Mr. Thackeray gives statistics of the total proper motions of stars up to and including $9^{m}\cdot0$ on the scale of the *Bonn Durchmusterung* which lie between the limits of declination $+24^{\circ}$ and $+32^{\circ}$. The area of this zone

is about one-sixteenth of the whole sky, and the zone both cuts the Milky Way and passes near its pole. It thus gives a very good sample of the whole sky, and affords ample material for the investigation of the distribution of proper motions for stars of different magnitudes.

The numbers given by Mr. Thackeray have been rearranged so as to show the number of stars with proper motions above certain limits out of all the stars down to various limiting magnitudes. This rearrangement is shown in Table I.

TABLE I.

Number of Stars with Proper Motions above certain Limits.

Mag. / p.m.	$0^m-5^m\cdot9$.	$0^m-6^m\cdot9$.	$0^m-7^m\cdot9$.	$0^m-8^m\cdot5$.	$0^m-8^m\cdot9$.	$0^m-9^m\cdot0$.
>50″	4	8	19	27	31	33
>30	8	17	44	75	92	98
>20	17	34	84	148	196	215
>15	26	58	126	231	318	368
>10	45	129	289	514	724	868
> 5	77	348	776	1494	2273	2902
> 0	165	626	2195	5387	9123	12046

It will be seen from inspection that the figures in any column obey approximately the law $\frac{n}{N} = \frac{c^2}{c^2+s^2}$, where n is the number of stars with proper motions $>s$; N is the total number of stars, and c is a constant for each column.

TABLE II.

Values of c *calculated from formula* $n/N = c^2/(c^2+s^2)$.

Mag. / p.m.	$0^m-5^m\cdot9$.	$0^m-6^m\cdot9$.	$0^m-7^m\cdot9$.	$0^m-8^m\cdot5$.	$0^m-8^m\cdot9$.	$0^m-9^m\cdot0$.
>50″	7″·74	5″·66	4″·65	3″·53	2″·91	2″·61
>30	6·78	5·01	4·24	3·55	3·02	2·71
>20	6·78	4·79	3·91	3·35	2·96	2·70
>15	6·47	4·79	3·70	3·21	2·85	2·66
>10	6·12	5·09	3·91	3·24	2·94	2·78
> 5	4·68	5·59	3·70	3·10	2·88	2·82
Mean	6·46	5·16	4·02	3·32	2·93	2·71
>20 to >10	6·46	4·89	3·84	3·27	2·92	2·71

It will be seen that for the bright stars there are more large proper motions than are indicated by the formula, but as the range is widened the formula fits the observations very satisfactorily.

It will be shown that a formula of this character follows from the assumption that the stars whose apparent magnitude $> m$ are distributed in space according to the law $re^{-a^2h^2r^2}dr$. The number of stars in any region of the sky with velocities between u_1 and $u_1 + du_1$ in a direction perpendicular to the solar motion and to the line of sight is taken as proportional to $e^{-h_1^2u_1^2}du_1$. Similarly, $e^{-h_2^2(u_2-\mathrm{U})^2}du_2$ is proportional to the number with velocities between u_2 and $u_2 + du_2$ perpendicular to the line of sight and in the plane of the solar motion, where U is the resolved part of the solar motion in that part of the sky. The values of h_1 and h_2 differ in accordance with Schwarzschild's ellipsoidal hypothesis. The use of different values makes the analysis complicated, and as the formula aimed at is to apply to total proper motions over a large region of the sky, h_1 has been taken equal to h_2 and the subscripts dropped.

Putting $u_1 = v \sin\theta$ and $u_2 = v\cos\theta$ where v is the total velocity perpendicular to the line of sight, the distribution

$$re^{-a^2h^2r^2}dr\ e^{-h^2u_1^2}du_1\ e^{-h^2(u_2-\mathrm{U})^2}du_2$$

is the same as

$$re^{-a^2h^2r^2}dr\ e^{-h^2(v^2-2v\mathrm{U}\cos\theta+\mathrm{U}^2)}vdvd\theta.$$

Putting $v = sr$ where s is the total p.m., this becomes

$$r^3sdrdsd\theta\ \exp.\ \{-(a^2+s^2)h^2r^2 + 2h^2rs\mathrm{U}\cos\theta - h^2\mathrm{U}^2\}.$$

Expanding exp. $(2h^2rs\mathrm{U}\cos\theta)$ and integrating with respect to θ, this becomes

$$e^{-h^2\mathrm{U}^2}sdrds\sum_0^\infty \frac{h^{2n}\mathrm{U}^{2n}}{n!^2}s^{2n}\cdot e^{-h^2(a^2+s^2)r^2}\cdot h^{2n}\cdot r^{2n+3}\cdot dr.$$

Integrating next with reference to r, the number of proper motions between s and $s+ds$

$$= \frac{sds}{(a^2+s^2)^2}e^{-h^2\mathrm{U}^2}\left\{1 + \frac{2}{1!}y + \frac{3}{2!}y^2 + \ldots\right\}$$

where

$$y = \frac{h^2\mathrm{U}^2s^2}{a^2+s^2}.$$

This

$$= \frac{sds}{(a^2+s^2)^2}\left(1 + \frac{h^2\mathrm{U}^2s^2}{a^2+s^2}\right)\exp.\left(-\frac{a^2h^2\mathrm{U}^2}{a^2+s^2}\right).$$

Integrating with reference to s, the proportion of total proper motions less than s

$$= \frac{s^2}{a^2+s^2}\cdot\exp.\left(-\frac{a^2}{a^2+s^2}\cdot h^2\mathrm{U}^2\right).$$

This formula may be compared with the formula $\frac{\tau}{\sqrt{(\tau^2+a^2)}}$ for

the proportion of the resolved proper motion $<\tau$ in a direction perpendicular to a stream velocity U, and the proportion $\frac{v}{\sqrt{v^2+a^2}}$ $-\frac{a^2}{a^2+v^2}h^2U^2$ between $+v$ and $-v$ in the direction of a stream velocity U. (*M.N.*, vol. lxxiii. pp. 337 and 403.)

If $\bar{v}$ be the mean peculiar velocity in one co-ordinate $h\bar{v}=\frac{1}{\sqrt{\pi}}$, thus $hU=\frac{1}{\sqrt{\pi}}\frac{U}{\bar{v}}$; $U=V\sin p$ where V is the velocity of the solar motion and p the angular distance from the apex. It will be found that $\bar{v}$ is not very different from the solar motion V.

Thus h^2U^2 lies between 0 and ·3. The exponential may accordingly be expanded, and the formula for the proportion of total proper motions less than s is

$$\frac{s^2}{a^2+s^2}\left\{1-\frac{a^2}{a^2+s^2}\cdot h^2U^2+\frac{1}{2!}\left(\frac{a^2}{a^2+s^2}\right)^2\cdot h^4U^4-\text{etc.}\right\}.$$

This is approximately equal to

$$\frac{s^2}{s^2+a^2(1+h^2U^2)}.$$

The difference between this and the more exact form is

$$\frac{s^2}{a^2+s^2}\cdot\tfrac{1}{2}\cdot\left(\frac{a^2}{a^2+s^2}\right)^2h^4U^4.$$

The maximum value of this for $s=a$ is $\frac{1}{16}\times\frac{1}{9}$ or $\frac{1}{144}$ of the total number of stars, or $\frac{1}{72}$ of the number with proper motion $>s$.

For the zone $+24°$ to $+32°$ the mean value of U^2 is ·42V^2. Thus the value of c^2 found from the observations may be equated to $a^2+\frac{\cdot42}{\pi}\frac{V^2}{\bar{v}^2}$ or $a^2+\frac{\cdot42}{\pi}M^2$ where M is the parallactic motion.

Provided that M and a are proportional, the quantities c of Table II. are proportional to the mean parallaxes of the different groups of stars. Using the means in the last line (*i.e.* the quantities obtained from stars $>20''$, $15''$, and $10''$ per century, omitting those >50 because of the small numbers on which they depend, and those $>5''$ because accidental errors in the p.ms. will affect the figures appreciably), and the total numbers of stars between certain limits of magnitude given in Table I., quantities are found which are proportional to the parallaxes of stars $6^m\cdot0$–$6^m\cdot9$, $7^m\cdot0$–$7^m\cdot9$, $8^m\cdot0$–$8^m\cdot5$, $8^m\cdot6$–$8^m\cdot9$, and $9^m\cdot0$.

For example, for 165 stars	$0-5^m\cdot9$	CM =	$6''\cdot46$.
,, 626 ,,	$0-6^m\cdot9$	CM =	$4''\cdot89$.
Therefore ,, 461 ,,	$6^m\cdot0-6^m\cdot9$	CM =	$3''\cdot85$.

Thus we find

TABLE III.

Limits of Mag. B.D.	Mean Mag. H.C.	Value of C×M.	Parallactic Motion. M.	Ratio. C.
m m	m	"	"	
6·0 – 6·9	6·7	4·34	3·85	1·13
7·0 – 7·9	7·6	3·42	3·00	1·14
8·0 – 8·5	8·4	2·87	2·40	1·20
8·6 – 8·9	9·0	2·41	2·04	1·18
9·0	9·4	2·05	1·82	1·13

where the values of M are derived from the formulæ $M = 410\varpi$ and $\log \varpi = -1{\cdot}224 - {\cdot}120m$, given in the *Monthly Notices* for November. The mean value of C is 1·156.

Thus

$$a^2 + \frac{{\cdot}42}{\pi} M^2 = 1{\cdot}156^2 M^2$$

or

$$a = 1{\cdot}10M,$$

and

$$c = \sqrt{\left(a^2 + \frac{{\cdot}42}{\pi} M^2\right)} = 1{\cdot}05a.$$

The constancy of the values of C given in Table III. is sufficiently good. The value $a = 1{\cdot}10$ M gives a larger value for the peculiar motions as compared with parallactic motion than is given by line-of-sight observations. It agrees with the results obtained from the Carrington stars when allowance is made for the direction of the axes of Schwarzschild's ellipsoid with reference to the latter stars. This may be due to a defect in the assumed formula, which is derived from stars of moderately large proper motion, and may not suit those less than 5″ per century so well; but, on the other hand, it is possible that the peculiar velocities of the bright stars which have been determined from spectroscopic observations may be smaller than those of stars of larger apparent magnitude.

For the distribution of stars in space, our assumed formula is $n = Ne^{-a^2h^2r^2}$, where N is the total number of stars brighter than a certain limiting magnitude, and n is the number of these at a distance greater than r. The value of

$$ah = \frac{1}{\sqrt{\pi}} \cdot \frac{a}{\bar{v}} = \frac{1}{\sqrt{\pi}} \times \frac{1}{1{\cdot}10} \times \frac{a}{\bar{V}},$$

and measuring distance in parsecs, and proper motions in seconds per century, $\bar{V} = 410$.

Thus

$$ah = \frac{a}{1{\cdot}10 \times 410\sqrt{\pi}} = \frac{c}{1{\cdot}156 \times 410\sqrt{\pi}}$$

where c is the quantity given in Table II. and

$$\log n = \log N - 10^{-6} \times 1{\cdot}417c^2r^2,$$

from which the following table is derived:—

TABLE IV.

Limits of Magnitude. m m	Log n.	$N-n$ (r small).	r_0.	Limit.
0 – 5·9	$2\cdot2174-25\cdot7r^2\times10^{-6}$	$\cdot010r^2$	92	260
0 – 6·9	$2\cdot7966-14\cdot7r^2\times10^{-6}$	$\cdot021r^2$	121	436
0 – 7·9	$3\cdot3414-9\cdot07r^2\times10^{-6}$	$\cdot046r^2$	155	607
0 – 8·5	$3\cdot7313-6\cdot52r^2\times10^{-6}$	$\cdot081r^2$	182	757
0 – 8·9	$3\cdot9601-5\cdot25r^2\times10^{-6}$	$\cdot110r^2$	204	869
0 – 9·0	$4\cdot0808-4\cdot52r^2\times10^{-6}$	$\cdot125r^2$	219	953

The distance r_0 at which the number of stars in the shell between r and $r+dr$ is a maximum is given by $1-2a^2h^2r^2=0$. The corresponding values of r are given in the fourth column. The last column gives the extreme limit to which the stars of the several groups reach, measured in parsecs.

Putting $r=20$ in the third column, it is seen that there are, according to this formula, 4, 8, 18, 32, 44, 50 stars with parallax $>''\cdot05$ within these limits of magnitude in this part of the sky. For the whole sky the numbers should be multiplied by 16. For the brighter stars the numbers appear to be much too small. The number 50 down to $9^m\cdot0$ B.D. in this zone or 800 from the whole sky does not seem at all improbable. Out of 60 stars with proper motions $>20''$ per century measured for parallax at Greenwich, 12 or 1 in 5 were found to have parallaxes $>''\cdot050$. This proportion gives about 700 in the whole sky, leaving 100 for stars of proper motion less than $20''$ per century. The extreme limits of the stars as given by the formula are too close. For the brighter stars a better agreement with the observed numbers of proper motions could be obtained, following Professor Eddington, by use of two terms $n=N_1\frac{s^2}{c_1^2+s^2}+N_2\frac{s^2}{c_2^2+s^2}$, the constants being chosen so as to suit the large and small proper motions.

The distribution of the stars according to the formulæ already obtained are given in Table V.

TABLE V.

Distribution in Distance of Stars of different Magnitudes.

Parallax.	Distance.	$0-5^m\cdot9$.	$6^m\cdot0-6^m\cdot9$.	$7^m\cdot0-7^m\cdot9$.	$8^m\cdot0-8^m\cdot5$.	$8^m\cdot6-8^m\cdot9$.	$9^m\cdot0$.
$>''\cdot10$	<10	1	1	$2\frac{1}{2}$	$3\frac{1}{2}$	3	$1\frac{1}{2}$
$''\cdot100-''\cdot050$	10 – 20	3	3	$7\frac{1}{2}$	$10\frac{1}{2}$	9	$4\frac{1}{2}$
$\cdot050-\cdot025$	20 – 40	11	13	30	42	34	20
$\cdot025-\cdot010$	40 – 100	59	89	176	300	241	125
$\cdot010-\cdot005$	100 – 200	76	242	511	852	778	448
$005-\cdot003$	200 – 333	15	102	500	942	992	670
$\cdot003-\cdot002$	333 – 500	0	11	312	945	1361	1200
$\cdot002-\cdot001$	500 – 1000	0	0	12	115	317	450

The luminosity law was determined by dividing the stars into half magnitudes and determining the numbers between limits of distance whose logs progressively increased by ·1. The result obtained was

$$\log N \,.\, dM = \{5{\cdot}430 - {\cdot}0091(M - 17{\cdot}8)^2\}dM,$$

where M is the absolute magnitude. The result cannot be considered satisfactory, because the coefficient of M^2 obtained from values M = 0 to M = 5·0 was nearly $-{\cdot}02$, and nearly zero as found from 5·0 to 10·0.

List of Stars in Zone +0° 0′ *to* +5° 0′, *the Proper Motions of which appear to be large, and are not in Boss's or Porter's Catalogues.*

(*Communicated by the Astronomer Royal.*)

A comparison between the Albany Catalogue for 1875 and the Abbadia Catalogue for 1900 gave 7000 stars common to both. The following list gives those stars which appear to have large proper motions. The proper motions are only approximate and not corrected for systematic error, and are given to three places of decimals in Right Ascension and two in Declination.

List of Stars with apparently large Proper Motions (*not in Boss's or Porter's Catalogues*) *from a Comparison of Albany Catalogue* (*A.G.*, 1875) *and Abbadia*, 1900.

(These P.M.'s are only approximate, and not corrected for systematic errors.)

B.D. No.	Albany No.	Approximate R.A. 1900.	Approximate Dec. 1900.	Magnitude.	P.M. R.A. (s·001).	P.M. Dec. (″·01).
		h m s	° ′			
3°68	115	0 29 48	3 50	8·8	+ 4	−21
2·72	117	30 4	2 40	8·8	+ 6	−17
0·181	289	1 0 56	1 11	8·8	+10	−15
2·315	569	56 40	2 24	8·8	−10	−27
2·321	577	57 49	2 52	6·9	+11	+16
3·339	680	2 22 32	3 59	8·5	+ 8	+22
2·393	704	26 20	3 0	7·9	− 5	−20
1·543	886	3 1 13	1 36	8·9	+23	−85
4·519	941	11 30	4 40	8·8	0	−29
1·577	956	13 45	1 57	8·8	+ 5	−16
3·515	1076	35 57	3 13	9·1	− 5	−23
1·661	1110	41 30	2 1	9·1	+ 4	+17
0·659	1120	44 24	1 4	8·4	+13	−65
5·678	1332	4 28 32	5 11	8·2	− 8	−25

B.D. No.	Albany No.	Approximate R.A. 1900. h m s	Approximate Dec. 1900. ° ′	Magnitude.	P.M. R.A. (s·001).	P.M. Dec. (″·01).
2°816	1484	4 50 56	2 47	8·8	+ 8	−30
2·1041	1860	5 37 30	2 39	8·8	+13	−55
1·1677	2613	6 59 54	1 33	8·5	− 4	−27
4·1690	2812	7 19 44	3 57	8·9	− 7	−17
4·1714	2846	23 10	4 21	8·3	−13	− 4
4·1743	2893	28 18	3 59	8·8	−11	0
4·2025	3496	8 37 42	4 47	8·7	−11	− 6
4·2050	3548	44 0	3 51	7·6	−14	+10
4·2051	3549	44 9	4 3	8·2	−23	− 6
5·2143	3713	9 9 40	4 51	8·4	−11	+ 3
3·2300	3922	53 49	2 53	8·9	−12	−11
3·2331	3983	10 7 38	3 36	9·1	− 6	−19
3·2344	3997	11 46	2 55	8·6	−17	− 3
4·2510	4328	11 35 21	4 13	8·6	+ 9	−10
2·2516	4430	12 2 35	2 10	9·1	−16	−14
5·2592	4441	5 5	4 53	8·9	− 5	−31
2·2625	4647	13 2 27	2 42	9·0	+ 6	−20
3·2757	4692	14 4	3 26	8·0	−16	− 7
0·3065	4728	24 7	0 38	7·3	−15	− 3
3·2802	4765	33 24	2 50	9·0	−11	− 9
2·2783	4895	14 6 28	1 50	6·5	-12	+ 5
3·2896	4965	23 30	3 14	7·1	−16	+ 6
1·2959	5013	33 10	1 19	8·9	−13	0
3·2926	5030	37 54	3 7	8·8	−14	−11
4·3023	5231	15 27 1	4 24	8·7	−11	− 2
2·3157	5529	16 37 19	2 16	9·0	+ 1	−18
1·3347	5587	51 6	1 16	9·0	−15	+ 2
4·3288	5591	52 4	4 5	8·9	− 1	−14
4·3509	5924	17 42 51	4 59	8·7	−46	−21
4·3574	6049	59 37	4 26	8·8	−13	−26
4·4157	6750	19 29 38	4 21	8·9	− 4	+29
4·4165	6758	31 18	4 33	8·9	− 3	−23
3·4191	6905	50 4	3 48	9·2	-21	−22
2·4127	7067	20 13 59	2 54	8·6	+18	0
2·4203	7194	30 42	2 58	7·8	−12	− 5
5·4874	7615	21 44 8	5 15	8·8	+35	− 4
4·4762	7636	49 8	4 30	8·5	+16	+ 1
2·4453	7658	54 3	2 43	8·5	+13	+ 8
3·4691	7739	22 11 17	3 33	8·6	−11	−11
4·4879	7817	30 32	4 52	8·6	+23	−18

The Magnitudes of the Cordoba "Catalogo de las Zonas de Exploracion," −52° to −62°. By H. H. Turner, D.Sc., F.R.S., Savilian Professor.

1. The volume here dealt with is dated 1914, but was only received at Oxford in 1916. It contains about 90,000 stars, the places being given to $0^s{\cdot}1$ in R.A. and $0'{\cdot}1$ in Dec. and the magnitudes to $0{\cdot}1$ as far as $9{\cdot}9$; after that to quarter magnitudes as far as 12. Very few particulars are given in the all-too-brief introduction. It appears that the greater part of the work was done under Dr. Thome's direction with a 5-inch telescope, the object-glass of which was found broken by Dr. Perrine when he took over the directorship in 1909. He had the O.G. repaired and completed the work, "carrying out Dr. Thome's plans of work as far as they could be known," and has now earned our gratitude by publishing it. It will be seen from the following analysis that a few more particulars would be very welcome: especially valuable would be a list of the observers and the zones which they observed, with dates, for some of the anomalies noted below are probably due to personality.

2. The observations of magnitude are visual, and are therefore not comparable directly with the photographic magnitudes of the *Astrographic Catalogue*. Nevertheless if there is an "obscured patch" in the sky, or any equivalent systematic change in the relative numbers of bright and faint stars, it ought to be in evidence with both visual and photographic magnitudes. The main object of the present paper is to obtain some information about the course of the "obscured spiral" indicated in former papers, in a region of the sky where we have as yet no astrographic measures and seem likely to be without them for some time, according to such information from the Sydney Observatory as is forthcoming.

3. The method of counts under each heading of the magnitude scale is of course just as applicable to visual as to photographic records. Zones $-52°$, $-53°$, $-54°$, $-60°$, $-61°$ were all counted in full detail, in order to test the magnitude scale as fully as possible. But it will suffice to give here part of one of these zones only. For magnitudes up to $9{\cdot}9$ the general run of the different hours seems to show that anomalies of estimate are small, but from 10 onwards there are serious discordances, as shown in Table I., which gives the counts of fainter magnitudes in Zone $-52°$.

TABLE I.

Counts of fainter magnitudes in Zone −52°.

Mag. / R.A.	9·8.	9·9.	10.	10¼.	10½.	10¾.	11.	11¼.	11½.	11¾.	12.
0h	3	2	22	25	37	35	29	2	2	0	0
1	5	5	16	11	25	29	23	9	30	9	0
2	6	7	24	21	14	16	11	12	14	5	0
3	21	13	107	5	5	2	8	0	0	0	0
4	8	11	30	29	41	24	19	0	0	0	0
5	13	11	69	33	46	47	41	0	0	0	0
6	15	17	170	20	22	6	21	0	0	0	0
7	23	16	249	10	16	4	8	0	0	0	0
8	25	13	48	42	77	63	98	0	1	0	0
9	30	12	61	71	116	114	144	15	20	4	4
10	50	17	43	63	91	48	53	12	12	1	1
11	38	46	85	88	124	59	54	5	2	0	0
12	18	16	50	64	109	84	96	9	6	0	0
13	14	11	38	36	66	68	93	26	24	1	4
14	12	11	27	35	66	77	108	18	32	5	0
15	26	10	69	61	102	106	147	34	24	1	0
16	28	22	59	67	89	75	112	57	35	1	0
17	14	10	36	38	57	69	110	43	19	0	0
18	25	7	117	16	42	31	73	19	17	0	0
19	24	26	247	1	5	3	0	1	0	0	0
20	9	23	180	7	11	18	13	11	19	2	0
21	11	10	37	18	24	44	47	22	28	3	0
22	14	7	15	17	25	26	32	26	23	3	0
23	6	4	19	20	49	27	41	3	5	0	0
Totals.	438	327	1818	798	1259	1075	1381	324	313	35	9

4. It is clear that the count in 19h, for instance, is conducted on a different plan from that in 21h. Probably a different observer was at work, but without knowing details we can only regard the comparative counts as of no value. The only question is whether, as found in some other cases, there are certain points in the scale which can be trusted, although in between them the scale becomes untrustworthy. To examine this point, the cases in zones −52°, −53°, and −54° were collected where the number of stars registered as 10 is specially in excess. These are shown in Table II. First the actual counts are given, those under head 9·9 denoting all the stars with mags. 9·9, 9·8, 9·7, and 9·6, greater than the previous heading 9·5. For 10·0 the numbers represent the single heading 10; under 11·0 are given all with 10¼, 10½, 10¾, and 11.

5. Adding up the various columns we form the cumulative total N at the foot; and then log N is compared with a similar quantity derived from *all* the hours in these three zones. It will be seen that the difference is nearly constant, except for the heading 10·0, where there is a sudden jump. It is inferred, with some probability, that the undoubted anomaly in the scale for certain hours will not affect the comparison, through log N, of the points "All" and 9·9, or brighter. But it would be more satisfactory if these anomalies could be traced to their source.

TABLE II.

Cases of excessive 10 *counts in Zones* $-52°$, $-53°$, *and* $-54°$.

Mag. / R.A.	6·0.	7·0.	8·0.	8·5.	9·0.	9·5.	9·9.	10·0.	11·0.	All.
					Zone $-52°$.					
3^h	0	0	7	6	13	31	48	107	20	0
6	2	5	11	8	20	38	52	170	69	0
7	2	5	12	14	16	39	68	249	38	0
18	1	2	15	10	24	45	60	117	162	37
19	0	0	10	10	15	35	73	247	9	1
20	2	2	10	11	13	24	52	180	49	32
					Zone $-53°$.					
2	1	1	6	13	8	24	28	71	52	17
7	0	2	10	7	21	49	74	236	22	0
18	1	1	10	14	14	34	60	139	185	2
19	1	3	8	16	13	34	54	118	107	27
20	0	0	8	11	13	32	47	115	81	21
21	2	1	8	13	5	29	57	112	48	25
					Zone $-54°$.					
6	1	2	8	9	21	30	44	157	73	0
7	0	4	13	13	16	33	84	174	65	0
18	0	1	8	12	15	54	76	263	58	4
Total.	13	29	144	167	227	531	877	2455	1038	166
N.	13	42	186	353	580	1111	1988	4443	5481	5647
Log N.	1·11	1·62	2·27	2·55	2·76	3·05	3·30	3·65	3·74	3·75
All.	1·28	1·81	2·48	2·77	3·02	3·30	3·54	3·71	3·97	4·00
Diff.	·17	·19	·21	·22	·26	·25	·24	·06	·23	·25

6. The importance of the point represented by 9·9 is that, for the remaining zones, we need not count in detail in order to watch the behaviour of the difference

$$\log (\text{All}) - \log 9{\cdot}9.$$

The total number of stars, "All," is given for each hour by the running number; and it is easy to pick out those brighter than 10 by their having a decimal of magnitude, which catches the eye. The results for all the ten zones are given in Table III. The constant shown at the foot of each column has been subtracted from the differences, so as to reduce each column to mean value zero.

7. The phase of the first harmonic has in the last few papers (on the astrographic magnitude scales) been adopted as the reference point for tracing the backbone of the spiral, the effect being regarded as an "obscuration."

In Table IV. are given, first, the coefficients A and B (of sin θ and cos θ) as calculated directly. In the first column the declination is increased by 0°·5, since each zone is only one degree wide and not two degrees, as for the *Astrographic Catalogue*.

TABLE III.

Values of log All − log 9·9 for different Zones.

(The unit is 0·01).

	−52°.	−53°.	−54°.	−55°.	−56°.	−57°.	−58°.	−59°.	−60°.	−61°.
0h	+ 5	−11	− 1	+ 1	− 5	− 1	+ 4	+ 6	+ 1	+14
1	+ 9	+ 2	+ 2	+ 5	− 1	+ 2	+11	+12	+15	+ 8
2	− 6	− 1	− 5	− 3	+ 8	+ 4	+ 9	+ 6	+12	+13
3	−13	− 6	− 3	+ 2	+ 4	+11	+ 2	+ 4	+22	+22
4	− 8	− 4	−10	− 8	0	+ 8	+ 8	+ 4	+ 9	+18
5	+ 5	− 5	− 8	+ 5	+ 9	+ 3	0	+ 3	+15	+21
6	− 4	− 3	+ 2	+ 3	− 3	+ 2	− 3	+ 3	+ 2	+17
7	− 2	− 3	− 7	− 2	− 3	− 1	0	− 1	− 1	+11
8	− 3	+ 1	+ 8	− 6	+ 6	+14	+16	+ 1	+12	+11
9	+ 2	+ 5	− 8	−10	− 6	− 4	0	+ 9	+ 9	+ 4
10	−13	− 3	+ 1	+ 1	−12	−10	−10	−14	−19	− 7
11	− 7	− 6	− 9	− 6	− 7	−16	−12	−18	−15	−20
12	+ 3	+ 5	+ 8	− 2	− 9	+ 2	+ 2	0	− 4	−22
13	− 1	+ 3	− 2	+12	− 6	− 4	+ 4	− 3	− 3	−14
14	+12	+ 3	− 2	+ 1	− 3	− 4	0	− 5	− 6	− 8
15	+ 6	− 4	+ 7	+ 8	+ 4	− 7	−24	− 9	+ 3	0
16	− 1	+ 6	+ 4	− 1	− 3	− 7	−18	− 6	− 3	+ 8
17	+ 4	+ 4	+ 7	− 1	+ 3	−17	−27	−14	−21	−21
18	− 1	+ 9	+ 1	+ 3	+ 2	+ 4	+ 8	−10	−21	−16
19	− 4	+ 3	+ 5	− 2	− 3	+ 5	+11	+ 6	− 5	−13
20	+ 3	+ 3	+ 1	+ 4	+ 9	+14	+18	+11	+19	+24
21	+ 3	− 2	+ 4	0	+12	+ 3	+ 6	+12	+10	−21
22	− 1	+ 4	− 3	0	+ 2	+ 4	+ 6	+10	− 5	−20
23	+ 2	− 5	+ 6	− 3	+11	− 2	−14	− 7	−21	− 7
Const.	48	44	46	44	40	41	38	39	34	28

TABLE IV.

Particulars of "Obscuration" for the Zones.

(The unit for A, B, and R is 0·01).

Dec.	A.	B.	R.	θ_0.	ψ_0.
−52°5	− 3·3	+0·3	3·3	96°	+ 17°
−53·5	− 3·4	−1·8	3·7	63	− 20
−54·5	− 3·6	+0·3	3·6	95	+ 9
−55·5	− 2·0	+0·2	2·0	95	+ 5
−56·5	− 0·6	+5·6	5·7	174	+ 81
−57·5	+ 3·2	+5·0	5·9	212	+115
−58·5	+ 3·9	+5·5	6·7	215	+114
−59·5	+ 2·9	+6·9	7·5	203	+100
−60·5	+ 7·8	+5·0	9·2	238	+131
−61·5	+12·7	+5·3	13·8	248	+137
−65	− 3·5	−4·9	6·0	36	[+271]

Next the coefficient R, where $R^2 = A^2 + B^2$. There is a distinct increase in R with declination in the Cordoba results. The results for −65° are quoted from the note on the Melbourne Counts given in *M.N.*, lxxvii. p. 40, being determination (*a*), which is obtained by a comparison of columns for magnitude 13·20 and 12·25 on the C. and M. scale. This scale is only doubtfully applicable to the Cordoba Visual Counts (see below), but assigns 12·3 and 11·3 as the magnitudes compared, the interval being thus close to 1·0 magnitude as at Melbourne, though the stars used are about one magnitude brighter.

Next is given θ_0, the R.A. of greatest obscuration. If we write

$$A \sin\theta + B \cos\theta = -R \cos(\theta - \theta_0)$$

we see that $\tan(\theta_0 - 180°) = A/B$.

8. In previous papers it has been suggested that the values of θ_0 indicate a spiral

$$\psi_0 \equiv \theta_0 + 3{\cdot}66\delta - 247° = 0,$$

where δ is the declination of the zone analysed. The values of ψ_0 are near zero at first, but then suddenly increase. The result for Melbourne suggests that the increase is maintained, though it is unprofitable to attempt an exact formula at present. The equation to the spiral is, of course, only intended as a rough approximation, and since the spiral is probably not related to our Earth's axis, the equation is clearly liable to modification as we approach high values of δ, of either sign. But without further confirmation we cannot exclude also the possibility of systematic error of some kind.

9. One piece of evidence, of a reassuring kind, appears at the other end of the zones where they join on to the Cape astrographic zones. In Table V. are given all the consecutive determinations

of A and B. Accidental error being clearly rather large, means of every consecutive three zones have been taken, and from these means smoothed values of $-R$ and θ_0 have been determined. It will be seen that the junction with the Cape is quite satisfactory.

TABLE V.

Juxtaposition of Cape and Cordoba series.

δ.	A.	B.	A′.	B′.	R_0'.	θ_0'.	ψ_0.
−41°	− 5·4	−4·6					
−42	− 9·7	−3·7	−4·4	−5·1	6·7	41°	0
−43	+ 1·8	−6·9	−4·1	−4·5	6·1	43	− 2
−44	− 4·3	−2·8	−2·2	−4·4	4·9	27	− 21
−45	− 4·2	−3·3	−3·9	−4·0	5·6	45	− 7
−46	− 3·3	−6·0	−5·7	−5·5	7·9	46	− 10
−47	− 9·7	−7·3	−6·9	−6·6	9·5	46	− 13
−48	− 7·6	−6·6	−7·9	−4·1	8·9	63	0
−49	− 6·3	+1·7	−5·4	−2·2	5·8	68	+ 1
−50	− 2·4	−1·7	−2·6	−0·9	2·7	71	+ 1
−51	+ 0·9	−2·6	−1·6	−1·3	2·1	53	− 21
−52·5	− 3·3	+0·3	−1·9	−1·4	2·4	55	− 24
−53·5	− 3·4	−1·8	−3·4	−0·4	3·4	83	0
−54·5	− 3·6	+0·3	−3·0	−0·4	3·0	82	− 4
−55·5	− 2·0	+0·2	−2·1	+2·0	3·0	132	+ 42
−56·5	− 0·6	+5·6	+0·2	+3·6	3·6	183	+ 90
−57·5	+ 3·2	+5·0	+2·2	+5·4	5·8	200	+103
−58·5	+ 3·9	+5·5	+3·3	+5·8	6·7	210	+109
−59·5	+ 2·9	+6·9	+4·9	+5·8	7·6	220	+117
−60·5	+ 7·8	+5·0	+7·8	+5·7	9·7	234	+127
−61·5	+12·7	+5·3					

There is one element of uncertainty at the junction. The differences of log N are taken for two magnitudes separated by about a unit magnitude in each case, but the separation is not very accurately determined. It should be consistently the same throughout each of the two series, but the Cape value may differ a little from the Cordoba value. It will be seen that this would make a misfit in R, but would not affect θ and ψ. At present attention has been chiefly concentrated on θ and ψ, which trace out the backbone of the suggested spiral ; and only approximate estimates, on which no great stress is laid, have been made of the intensity R, so that there may be a misfit in R of the kind indicated. But the run of the numbers shows that it cannot be serious. There is a drop in R to the junction, and then a rise, but the drop and rise are clearly indicated within the two separate series of results, which are on scales consistent within themselves.

10. The values of θ and ψ are sensibly unaffected by any such change of scale, which could only assign slightly different weights to the three components, of which the mean is taken as a smoothing process: and we see that the values of ψ, which should be zero if the suggested equation for the spiral, viz.

$$\psi = \theta + 3{\cdot}66\delta - 247^\circ = 0,$$

is absolutely correct, are satisfactorily small as far as -54° declination. Thus not only do the Cordoba results fit satisfactorily on to the Cape results, but they confirm the spiral as far as -54°. After that there is a comparatively sudden increase in ψ. To keep it near zero we must largely increase the coefficient (3·66) of δ. This coefficient indicates the angle M at which the spiral cuts the meridians, the value of M corresponding to $\tan M = 3{\cdot}66$ being $74^\circ{\cdot}7$. To fit the portion from $\delta = -54^\circ$ to $\delta = -60^\circ$ we must have $\tan M = 20$ or 30, *i.e.* $M = 87^\circ$ or 88°, so that the spiral is only very slightly inclined to a parallel of declination. This would afford a natural explanation of the increase in the coefficient R, for the backbone of the spiral, *i.e.* the region of greatest obscuration, would remain longer in close contact with the zone under analysis as we reach higher declinations.

11. We may now examine the distribution of the numbers in Table III. according to ψ; as was done for the Cape results on p. 153 of *M.N.*, lxxvi. For this purpose the different zones were started as below.

Dec.	-52°	-53°	-54°	-55°	-56°	-57°	-58°	-59°	-60°	-61°
ψ_0	4^h	5^h	6^h	9^h	12^h	13^h	14^h	15^h	16^h	17^h

The means of the zones thus arranged are given in Table VI., those for the Cape being repeated (from the above reference) for comparison.

TABLE VI.

ψ.	Cord.	Cape.	ψ.	Cord.	Cape.	ψ.	Cord.	Cape.	ψ.	Cord.	Cape.
h			h			h			h		
(0)	−6·7	−3·0	(6)	+2·2	−1·3	(12)	+4·4	+7·2	(18)	−2·3	−2·6
(1)	−8·1	−1·2	(7)	+1·5	+1·5	(13)	+5·9	+3·8	(19)	−5·5	−6·2
(2)	−8·1	−2·3	(8)	+2·3	+2·3	(14)	+4·3	+7·8	(20)	−3·7	−6·3
(3)	−3·2	−0·7	(9)	+4·8	+1·9	(15)	+2·9	+8·8	(21)	−3·1	−4·0
(4)	+0·7	−0·3	(10)	+6·0	+2·7	(16)	+1·8	+2·4	(22)	−5·9	−6·5
(5)	+1·0	−3·1	(11)	+9·0	+5·4	(17)	+2·4	−3·8	(23)	−2·6	−3·4

12. There are systematic differences between the two sets, a large part of which can be represented by the second harmonic

$$3{\cdot}0 \cos (\psi - 16^h),$$

but there is no reason why the two sets should be the same. It is a little disappointing not to find better marked in the Cordoba results the "sharp edge" on the following side of ψ, noticed for

the Cape, and to some extent for Melbourne (*M.N.*, lxxvii. p. 41). Between $\psi = 15^h$ and $\psi = 19^h$ the Cape results drop from $+8{\cdot}8$ to $-6{\cdot}2$, the Cordoba results only from $+2{\cdot}9$ to $-5{\cdot}5$. Perhaps the edge is softened by the spiral clinging closer to the zones of declination.

13. We may remark how little influence the galaxy seems to have on these results. Any influence it has should appear as a first harmonic with either maximum or minimum steadily near 12^h, since the galaxy reaches its greatest declination south at $12^h - 60°$, and for smaller declinations the two points of intersection with the zone are symmetrically placed with respect to 12^h, and are not far from it. But we have seen that the maximum obscuration, so far from remaining sensibly constant for these zones, changes very rapidly with the declination. It is difficult to see how such a phenomenon could be related to the galaxy: it can certainly have nothing to do with galactic latitude, though possibly it might be connected with galactic longitude.

14. Another point of interest is the march of the obscuration according to magnitude. Taking the three zones $-52°$, $-53°$, and $-54°$, which were counted in full detail, and adding together the portions 12^h–22^h, and then the portions 23^h–11^h, we get the following results for log N for different magnitudes.

Table VII.

Log N for "obscured" and "unobscured."

Mag. (Cordoba)	6·0	7·0	8·0	8·5	9·0	9·5	9·9	All
Mag. (C. & M.)	6·2	7·2	8·5	9·2	9·7	10·4	11·0	12·2
12^h–22^h	1·342	1·909	2·610	2·932	3·170	3·461	3·696	4·185
23^h–11^h	1·556	2·065	2·695	2·967	3·201	3·493	3·732	4·161
"obs."–"un."	+·214	+·156	+·085	+·035	+·031	+·032	+·036	−·024

The change with magnitude is thus far from uniform throughout. It is nearly uniform at 0·070 per magnitude from magnitude (Cordoba) 6·0 to 8·5. Then from 8·5 to 9·9 it is sensibly constant; and then in the next 1·2 mag. (on scale C. & M.) it falls ·055. This intermittence in the phenomenon seems scarcely compatible with the idea of *obscuration*, which has been provisionally adopted as an explanation: but nothing essential turns on this particular hypothesis.

15. The C. and M. magnitudes in Table VII. are taken from the table given on p. 145 of *M.N.*, lxxv., reducing the hourly counts in zones of 1° to four square degrees by the factor $4/15 \cos\delta$.

16. To throw further light on the march of the obscuration the first harmonics were calculated for various magnitudes, using the sum of zones $-52°$, $-53°$, and $-54°$. The results are as shown in Table VIII.

TABLE VIII.

Zones $-52°$, $-53°$, *and* $-54°$. *First harmonics of log N.*

Mag.				Means.		Differ. Mag.	
Cord.	C. and M.	A.	B.	A.	B.	A.	B.
6·0	6·2	+7·0	−11·5				
7·0	7·2	+4·7	−17·3	+3·2	−14·2		
8·0	8·5	+1·0	−13·7				
8·5	9·2	−3·3	−15·8			−4·0	−3·4
9·0	9·7	−3·5	−19·2				
9·5	10·4	−4·3	−20·9	−4·1	−20·9		
9·9	11·0	−4·5	−22·7			−3·4	−1·9
All	12·2	−8·2	−23·2	−8·2	−23·2		

17. Here, again, we see the curious halt, in both A and B, from mags. 9·0 or earlier to 9·9. For the bright stars the determinations of A and B are of course subject to greater uncertainties, owing to the smallness of the available numbers; indeed, for the first group (mag. 6·0 Cordoba) the determination is rendered theoretically impossible, since one of the twelve terms submitted to analysis contains no stars at all (3^h and 4^h). For purposes of computation one star was assumed in this group, making the logarithm zero; but this is of course not a strictly legitimate process.

18. Taking the means of A and B for the first three groups, we may consider that they correspond to magnitude 7·3 (C. and M.). The slopes in A and B are fairly steady down to somewhere between 9·2 and 9·7 (C. and M.), say 9·3, so that the differences correspond to a difference of two magnitudes. This is only rough, but gives the numbers −4·0 and −3·4 as the change per mag.

19. The slope then disappears from mag. 9·3 to 11·0 (C. and M.). If we assume that it then reappears from 11·0 to 12·2, the differences divided by 1·2 give us −3·4 and −1·9, nearly the same numbers as before. This representation by two similar slopes separated by a horizontal line is probably only an approximation, but exhibits the main features of the change.

SUMMARY.

1. Detailed hourly counts of the magnitudes of the Cordoba zones $-52°$, $-53°$, and $-54°$ show that after magnitude 9·9 the magnitude scale is quite untrustworthy; but the limit indicated by the faintest stars included is satisfactorily uniform.

2. Using the points "9·9" and "All" for the whole of the ten zones ($-52°$ to $-61°$) the differences of log N are found to vary round the circuit of each zone in a manner similar to that found elsewhere. The junction with the Cape Astrographic Counts ($-42°$ to $-51°$) is very satisfactory, carrying on the suggested "spiral" to $-54°$. After this, however, the spiral becomes suddenly much flatter.

The Irregular Variable RS Camelopardalis.

By H. C. Plummer, M.A.

Professor Turner's criticism of the paper on this subject by C. Martin and H. C. Plummer deals entirely with the discussion of the observations, and not with the observations themselves. As the responsibility of one author alone is thus involved, it will be convenient for him to make a short reply. Professor Turner's practical experience in applying periodic analysis has been so extensive that his views certainly deserve every respect.

He takes an early opportunity of remarking that "nothing here said challenges the analysis" of our paper. This is eminently reassuring, for it covers no less than thirteen out of fourteen sections of the paper. As a matter of fact, the final paragraph was added as an after-thought, and it may be that its brevity is partly responsible for a misunderstanding of its meaning. The criticism turns very largely on this unfortunate paragraph, and even on a single sentence, which is quoted no less than three separate times. Let it be said generally that, so far as Professor Turner's remarks will act as a warning against *any* too dogmatic deduction from the results of analysis of this kind, they have the writer's fullest sympathy and approval. On the other hand, it seems clear that he is disappointed that the star does not conform to a certain ideal type of variation. In that the blame must be placed on the star, not on us. The particular pitfall which Professor Turner describes in such pathetic terms could never have the slightest allurement for anybody who was seriously investigating the variation of the star. The one thing certain from the beginning was that a single ruling period, however long, was most improbable, and that such an assumption must be absolutely avoided in the representation.

It is a little hard to see how the spirit of this final paragraph could have been more completely misrepresented. In fact, its simple argument has been exactly inverted. With every allowance for any possible obscurity, it is obviously directed against the representation by means of an Argelander-Chandler formula. It suggests a more satisfactory alternative, the analysis of the recorded variation as a whole. But unfortunately, as it now appears, it showed that a colourable imitation of the Argelander-Chandler formula could be forced out of the more complete representation. The object was to exhibit the relations between the different methods, obviously not to support the one which is held to be inferior and which, according to Professor Turner, has never proved satisfactory. With this explanation the details of the adjusted formula, which in any case apply strictly only to this particular star, are seen to be of trivial importance. They do not deserve the critical attention which they have received.

This explanation also covers the phrase "a simple oscillation in the period 88·7 days," which obviously cannot be taken out of its

context. Is it reasonable to suppose that after taking the trouble to disentangle three periods (whatever their meaning may be), the work has been virtually thrown away, and any single period has been nominated in substitution? It is disappointing, after pointing out a means of escape from an old rut, to be brought back to it and to be told that it is a bad one.

To deny any intention of returning to that particular path is to render a considerable part of Professor Turner's criticism irrelevant. But that is not to say that the path actually followed is necessarily the right one. Here, naturally, there must be much latitude of opinion. A simple aggregate of four harmonic terms has been suggested for the variations of short period. Presumably, three questions are now involved: (1) How far is the representation of the observed past satisfactory? (2) With what probability can it be expected to extend to the future? (3) What physical reality lies behind its precise form? With regard to (1), Professor Turner is good enough to speak of "the beautiful accordance" between the observations and the formula. This is over-generous, but it may be taken to dispose of the first question. But this covers the modest aim of the main inquiry. To represent the past behaviour of the star at all did not prove the easiest of tasks, though that of course may argue lack of skill. It is, however, to be remarked that investigations of similar cases seem to have failed to reach even this preliminary stage.

When we come to the other questions a critic is in rather difficult case, for the simple reason that no attempt was made to decide them. The second is a question of probability which the future alone can settle. The representation may fail—perhaps one should say that it will very likely fail; and Professor Turner would probably be even more emphatic. But something may be added as to the reason. The critic may contend that the method is in defect: that another method would lead to more certain results. That is not believed. The failure will arise because the past record does not contain the whole of the required evidence, just as it is certainly defective in regard to the longer periodicities. In that case every method is doomed to definite failure, and the burden of the criticism is that an impossible task has been undertaken. Such criticism, in the present state of knowledge in this department, would surely carry its own condemnation.

The third question, important as it is, is even more remote. It was not even considered, and it seems idle to discuss it until the future has given its verdict on the preliminary issue. It will become profitable only when the permanent existence of the terms in question is assured, and the study of other stars has given a broader basis for discussion. In the meantime, it is not proposed to deal further with the argument contained in the middle of Professor Turner's note (§§ 6–11) beyond endorsing any warning against hasty conclusions, whether favourable or unfavourable to either side of the case. Above all, uncertain analogies are not in the least helpful and are to be deprecated.

Some reason for Professor Turner's misconception, and the difficulty of guarding against it by anticipation, may be seen in his description of the type of irregular variation shown by RS Camelopardalis. Apparently he regards it as a mild form of departure from regularity, with an approximate tendency to a regular period. Now from the beginning of the discussion such a view was never even contemplated. The irregularity of this star is so pronounced that so far from regarding it as a mere variant of the long-period regular variable, any affinity it may have with this class must be considered purely conjectural.

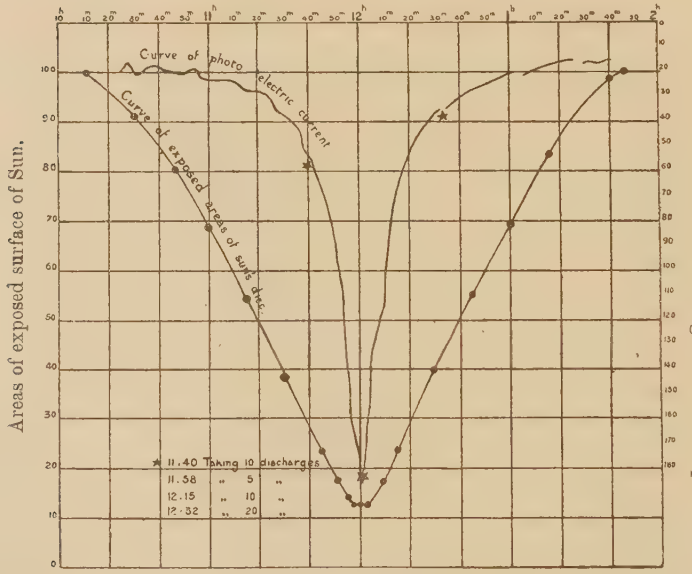
The possibility of making this reply at an early date is due to Professor Turner's courtesy in forwarding a copy of his note in proof.

Observations made at Adelaide during the Annular Eclipse of the Sun on 1916 *July* 30. Communicated by G. F. Dodwell.

The central line of the eclipse passed from the N.W. to the S.E. of South Australia, the eclipse being annular for a distance of about 75 miles on each side of the central line. The annular phase was well observed at Adelaide, approximately 40 miles S.W. of that line. The atmospheric conditions were all that could be desired, the sky in the vicinity of the sun remaining clear throughout the eclipse. A series of photographs was obtained at the Observatory with the 8-inch Cooke equatorial refractor by Mr. H. B. Willsmore, and visual observations of contacts, etc., made with the finder of that instrument. The first contact was at approximately July 30 $10^h\ 12^m\ 42^s$ (Adelaide Standard Civil Time). This is considered fairly exact, although the Moon's edge was not noticed until it was just inside the Sun's disc. Allowance was therefore made for its rate of progress. There were two or three spots visible on the Sun's surface, and the edge of the Moon reached the centre of the largest of these at $10^h\ 28^m\ 55^s$ a.m. A.S.T. Diminution of light became noticeable before 11 o'clock, and was accompanied by a perceptible fall in the air temperature. At $11^h\ 57^m\ 22^s$ a.m. the eclipse became annular. Mid-eclipse was at $0^h\ 0^m\ 6^s$. From measurements on the photographs the area of the annulus was 12·63 per cent. The appearance of the Sun during this stage of the eclipse was most interesting and impressive. The light was much enfeebled, and it was possible to look at the Sun with the naked eye, but the irradiation was still so great that without a dark glass the body of the Moon could not be distinguished, the Sun's appearance resembling that of a small lake of molten silver. The sky was a very deep blue, and a slight tinge of purple colour was noticed in the clouds near the horizon. The light was estimated as about equal to that immediately after sunset on a clear winter day. The sunlight streaming through the foliage of trees cast beautiful rings of light and shade on the ground. Birds

and animals were not uniformly affected by the eclipse. Some apparently took no notice, others appeared uneasy or behaved as if they expected the approach of night. The annular phase ended at $0^h\ 2^m\ 50^s$, thus lasting $5^m\ 28^s$. Immediately before the commencement and end of this phase the phenomenon of Bailey's Beads was well seen, and a photograph taken at $11^h\ 57^m\ 12^s$ recorded an example of it. The receding edge of the Moon reached the middle of the largest sun-spot at approximately $0^h\ 12^m\ 30^s$. The last contact was well observed at $1^h\ 44^m\ 39^s$. Observations of the eclipse made by many observers in the city and country were received at the Adelaide Observatory. Mr. A. E. Messent, of the

Areas of exposed surface of Sun (total surface = 100), and measurements of photo-electric current during eclipse.


Curve of photo-electric current
Curve of exposed areas of sun's disc
Areas of exposed surface of Sun.
Times in seconds for 20 discharges of electroscope cell.
11.40 Taking 10 discharges
11.58 " 5 "
12.15 " 10 "
12·32 " 20 "

Observatory staff, obtained photographs at Angaston on the central line.

Professor Kerr Grant has supplied the following account of actinic measurements made by himself and Mr. R. S. Burdon at the Adelaide University:—

"Observations on the photo-electric power of the Sun's light were made throughout the duration of the eclipse by means of a photo-electric cell similar in construction to that described by Hughes, *Phil. Mag.*, xxv. pp. 679–682. The chief difference was that the tube was made of glass and not quartz, which means that the effect was limited to light of wave length greater than 3300 A.U. The current through the tube was measured by means of a flapping gold-leaf electroscope as described by Zeleny. This method had been found to give very consistent results when tested by using

the light of an electric lamp; and, except for some irregularities, almost certainly observational, at the commencement of the eclipse, gave a continuous variation throughout the eclipse. When compared with the variation of exposed area of the Sun's disc, the graph of photo-electric effect shows in a very marked way the decrease of actinic value of sunlight as we pass from the centre to the edge of the disc."

The following are the temperatures recorded by Mr. E. Bromley, of the Meteorological Bureau, during the eclipse:—

Time a.m.	Temperature. Shade.	Sun.	Time p.m.	Temperature. Shade.	Sun.
10.0	53·0	101·2	12.0	50·3	54·0
10.15	53·0	104·0	12.5	50·0	52·4
10.30	53·1	103·5	12.10	49·7	52·28
10.45	53·8	101·0	12.15	49·5	54·2
11.0	53·8	95·0	12.20	49·6	57·3
11.10	53·4	89·0	12.25	50·0	61·0
11.20	53·0	83·3	12.30	50·3	64·4
11.30	52·6	76·4	12.40	51·2	72·5
11.40	52·1	72·9	12.50	52·0	80·5
11.45	51·7	65·0	1.0	52·7	87·6
11.50	51·1	61·6	1.15	53·9	97·5
11.55	50·9	58·1	1.30	55·2	105·0
			1.45	55·7	108·8
			2.0	56·0	109·2

Light northerly wind; sky clear except for a few clouds near horizon.

Probable Motions in the Spiral Nebula Messier 51 (Canes Venatici) found with the Stereo-comparator. Preliminary Communication. By Dr. S. Kostinsky, Astronomer at the Central Observatory, Pulkovo.

1. The great nebula Messier 51 is the most typical object of this kind and the best illustration to the cosmogonical hypothesis of Chamberlin-Moulton. For a long time observers have been interested in the question of possible changes in its form.

Lord Rosse, who detected the spiral structure of the nebula early in 1845, doubted whether such an object could exist in statical equilibrium without internal movements (*Phil. Trans.*, 1850, p. 504). Therefore his assistant, J. Stoney, in 1849–50 measured the positions of some stars and knots of the nebula with regard to the centre. Later, R. Copeland repeated these measures with the great reflector of the same Rosse observatory

at Birr Castle; unfortunately these visual observations (very difficult on account of the faintness of the object) were not exact enough for the purpose.

From 1889 to 1904 Isaac Roberts obtained a series of excellent photographs of the nebula with his 20-inch reflector; some of them were measured by him, especially for the purpose of studying the possible changes of the single parts of the nebula, and also more completely by his widow, Mrs. Dorothea Roberts. Certain indications of such changes seemed to result (*Rivista di Astronomia*, 1910, pp. 31–41 and 62–79), but the authoress herself believes her materials and her means for the study of it insufficient to solve the question of minute displacements.

2. Beginning with the year 1896, I succeeded in obtaining with the great astrograph at Pulkovo 12 plates of the above-mentioned nebula with exposures from 47 minutes to 3 hours; on some of them are to be seen many faint and well-defined details, which are effaced on account of over-exposure on the photographs of Roberts and the later ones of Keeler. This gives rise to the hope that at a sufficient difference of epochs small movements of the single knots will be distinguishable, if they exist at all.

As a matter of fact, on comparing stereoscopically the first of my photographs with the last ones, taken in 1896 March and 1916 April, I remarked almost indisputable displacements of some characteristic knots lying on the spirals; also movements were noticed of many stars—in the nebula itself and in its neighbourhood—some of which are probably physically connected with it.

Preliminary stereoscopic measures on the above-mentioned couple of plates of 36 single knots showed me that their observed proper motions with regard to the centre of the nebula seem to have a systematic character in the different parts. They led me to the following preliminary conclusions:—

(*a*) On the outer spiral the motion proceeds as though in general the single parts were moving away from the centre, so that the spiral has a tendency to draw itself together in the direction opposite to the hands of a clock;

(*b*) On the contrary, on the inner spiral in its eastern part the motion towards the centre prevails, and if there is any tendency to draw together it is rather in the opposite direction, *i.e.* with the hands of a clock;

(*c*) In the mean the annual proper motions of the single knots and stars are of the order $0''{\cdot}04$–$0''{\cdot}05$, but vary within wide limits.

I hope that a detailed working up of my material will soon allow me to prove these preliminary results.

Pulkovo:
1916 *May.*

Occultations of Stars by the Moon observed at the Royal Observatory, Greenwich, in the year 1916.

(*Communicated by the Astronomer Royal.*)

Date and G.M.T. 1916.				Star and Phase of Occultation.	Magnitude.	Observer.	Aperture.
	d	h	m	s				inches.
Jan.	15	12	5	1·43	χ Tauri, D	5·3	A.C.	$2\frac{3}{4}$
	,,			1·14	,, ,,	,,	R.C.	$6\frac{3}{4}$
	,,			1·13	,, ,,	,,	F.J.	10
Feb.	7	9	27	22·57[a]	W.B. O., 579, D	6·5	A.C.	$2\frac{3}{4}$
May	8	11	31	25·58	d' Cancri, D	5·9	A.C.	$2\frac{3}{4}$
	,,			25·12	,, ,,	,,	W.	10
Sept.	15	15	22	18·06[b]	ϵ Arietis, R	4·6	A.C.	$2\frac{3}{4}$
	16	11	41	50·90	26 Tauri, D*	6·6	H.	10
	16	12	3	48·30[c]	27 Tauri, D*	3·7	H.	10
	16	12	23	35·76	28 Tauri, D*	5·2	H.	10
	16	12	55	5·21	27 Tauri, R	3·7	H.	10
Nov.	2	8	35	56·85	94 B Capricorni, D	5·7	A.C.	$2\frac{3}{4}$
	7	6	47	23·74	212 B Piscium, D	7·1	H.	10

Phenomena marked with an asterisk occurred at the Moon's Bright Limb; all other phenomena at the Dark Limb. The telescopes used are as follows:—

Aperture 10 in., Astrographic Equatorial (Guiding Telescope).
,, $6\frac{3}{4}$ in., Sheepshanks Equatorial.
,, $2\frac{3}{4}$ in., Portable telescope in Altazimuth Dome.

The initials H., A.C., W., R.C., F.J., are those of Mr. Hollis, Mr. Crommelin, Mr. Witchell, Mr. Cullen, and Mr. Jeffreys respectively.

Notes.

a. The star was too faint to hold steadily.

b. Clouds passing; the power employed was too low to show the duplicity of the star.

c. The observer noted that the count of seconds was doubtful.

Royal Observatory, Greenwich:
1917 *Jan.* 11.

Micrometrical Measures of Double Stars (*11th Series*) *and New Double Stars.* By Rev. T. E. Espin, M.A.

The measures in the following list have been made with the 24-in. Calver reflector, with the exception of a few stars measured in the early part of 1914. Of the following stars of *h* there are, as far as I am aware, no other measures :—

h 1014	*h* 1715	*h* 2557
1017	1765	2559
1021	1831	2564
1031	1932	2568
1385	1978	2803
1571	2516	2869

Two of *h*'s stars I have been unable to find :—

(1) *h* 1679 was searched for on 1916 December 15. It is certainly identical with *h* 1673, the R.A. being one minute too great, and the position angle being reversed.

(2) *h* 2593, searched for on 1916 April 22, 28, May 3, and not found.

Name.	R.A. 1900. h m	Decl. ° ′	P. °	D. ″	Mags.	Nts.	Date.	Year.	
+44°·4550	0 0·0	+44 40	235·1	20·37	6·5 9·2	2	·991	14	
A 110	0·6	42 6	122·1	1·77	9·2 9·3	2	·902	16	
Espin 153	11·2	40 44	242·2	2·37	9·5 10·6	3	·004	17	
h 1014	13·4	41 56	50·5	11·51	9·4 9·5	2	·939	16	
h 1017	14·8	41 59	263·1	16·43	9·8 11·2	2	·939	16	
h 1021	16·6	41 40	246·7	6·08	9·1 9·5	4	·950	16	
h 1978	24·9	43 36	223·2	5·61	9·5 9·7	4	·925	16	
h 1031	27·1	41 10	142·8	6·16	9·6 11·5	3	·973	16	
ρ 62	29·7	44 17	310·5	6·79	9·1 10·0	2	·961	14	
Espin 444	31·4	44 57	193·0	2·63	9·2 9·5	2	·843	14	
Espin 445	32·6	44 54	348·7	3·82	9·3 10·3	1	·865	14	
Σ 44	33·0	40 26	266·8	10·30	8·6 9·2	2	·961	16	
Espin 157	1 7·8	40 37	92·1	15·85	7·0 12·8	3	·978	16	AB
			48·0	19·68	13·2	2	·981		AC
Arg. vi. 3	8·9	42 23	288·9	11·60	8·8 9·0	2	·861	16	BC
			101·9	34·36	A = 8·3	2	·861	16	AB
Espin 4	25·2	43 6	97·1	3·18	7·9 8·8	2	·100	16	
+43°·322	28·8	42 45	94·8	44·38	8·6 8·8	2	·018	16	
A 1527	55·2	43 11	238·8	5·21	8·2 10·1	3	·079	16	
Espin 48	2 3·4	42 23	200·1	9·27	7·4 11·0	3	·102	16	
ρ 66	13·0	42 32	187·4	6·10	9·3 10·2	2	·891	16	

Name.	R.A. 1900. h m	Decl. ° ′	P. °	D. ″	Mags.	Nts.	Date.	Year.	
Ho 319	3 14·6	+45 1	46·4	11·74	7·0 11·5	1	·873	14	AB
			308·1	18·13	15	1	·873	14	AC
ν Persei	38·4	42 16	46·6	31·40	4·0 12·0	2	·152	16	
A 1543	48·3	42 36	182·0	4·29	9·0 9·3	2	·105	16	
Espin 13	4 38·2	43 37	216·1	15·35	7·2 13·9	2	·139	15	AB
			346·1	16·79	14·2	2	·139	15	AC
Σ 599	43·8	44 47	337·3	10·23	7·9 8·6	2	·940	14	AB
			65·7	14·74	C=13·7	2	·940	14	BC
Σ 613	51·6	43 59	103·2	15·99	8·5 9·0	3	·136	15	AB
			17·6	15·60	C=11·0	3	·136	15	BC
OΣ 189	8 7·9	43 20	292·3	4·21	6·6 9·7	2	·168	15	
Espin 18	20·2	42 26	236·1	12·62	8·3 9·2	2	·258	16	
Σ 1369	9 29·1	40 25	149·1	24·71	6·8 7·8	3	·276	16	
h 2516	57·4	40 4	98·5	6·93	11·4 11·6	2	·272	16	
Σ 1405 rej.	59·9	40 4	252·2	22·11	8·0 10·5	2	·293	16	
Σ 1430 rej.	10 19·8	41 25	268·0	21·99	8·0 12·0	2	·326	16	AB
			235·9	22·91	C= 9·5	2	·326	16	AC
h 2531	22·5	40 43	3·0	8·76	8·5 8·8	3	·261	16	
h 2534	27·4	40 58	326·2	22·24	5·0 11·8	2	·265	16	
OΣ 223 rej.	32·3	40 57	133·7	22·68	7·5 10·7	2	·312	16	
Σ 1475 rej.	43·7	41 55	199·6	23·80	8·0 11·1	3	·316	16	
h 2557	11 2·0	44 2	25·9	23·32	9·4 9·6	2	·321	16	
h 2559	3·0	43 3	266·8	14·72	11·7 12·1	2	·293	16	
h 494	7·5	40 44	149·7	29·45	9·4 9·6	2	·361	16	
h 2564	11·2	42 49	129·0	32·45	8·5 10·5	2	·301	16	
h 2568	16·4	44 10	244·0	24·03	11·0 12·5	2	·321	16	
h 2596	12 2·3	43 39	238·5	33·94	8·2 10·7	2	·373	16	
Σ 1606	5·7	40 27	325·8	0·91	6·8 7·0	4	·377	16	
Σ 1624	11·7	40 9	149·4	6·13	7·2 9·4	2	·345	16	
Σ 1638 rej.	19·3	43 36	280·5	8·28	9·0 10·5	2	·372	16	
+43°·2276	50·4	43 7	72·1	38·67	9·3 9·5	2	·345	16	
Espin 125	13 0·9	42 13	119·9	2·39	8·6 9·6	2	·395	16	
h 2639	1·6	41 28	160·4	31·61	7·8 11·0	3	·377	16	AB
			136·6	56·18	C= 9·2	3	·377	16	AC
			291·5	31·34		2	·406	16	CB
h 1244	49·2	42 41	129·2	7·04	7·5 11·0	1	·427	16	
Espin 625	15 17·7	44 40	251·6	1·64	9·5 9·6	3	·355	16	
Σ 1956	29·7	42 8	36·7	1·61	7·7 8·9	3	·505	16	
Espin 631	16 32·1	43 38	161·4	10·66	8·0 10·0	2	·512	16	
h 2803	57·3	40 34	263·9	18·11	9·6 11·6	3	·592	16	

Name.	R.A. 1900.	Decl.	P.	D.	Mags.	Nts.	Date.	Year.	
	h m	° ′	°	″					
Espin 634	17 0·8	+42 20	97·8	2·25	9·5 10·5	1	·556	16	
Espin 636	33·4	41 46	127·1	2·31	9·3 10·1	2	·577	16	
+43°·2774	34·2	43 12	154·1	9·12	8·8 8·9	2	·681	15	
ρ 18	18 3·4	42 12	174·3	15·58	9·3 10·1	2	·571	16	
ρ 5	8·6	42 43	108·8	14·75	9·4 10·0	3	·668	15	
ρ 6	8·8	42 46	286·4	10·70	9·2 9·8	3	·668	15	
Espin 473	10·0	42 49	161·0	11·63	9·0 10·5	2	·768	15	AB
			280·1	31·01	C= 9·0	2	·768	15	AC
Σ 2393	41·7	38 12	23·4	13·92	7·3 9·5	2	·748	14	
h 1385	19 16·4	43 51	205·7	6·59	9·7 12·7	2	·799	14	
h 2869	18·3	42 2	357·9	8·54	10·8 11·7	2	·577	16	
A 594	24·1	42 53	332·6	2·36	9·0 9·2	2	·753	15	
Espin 23	46·8	44 54	134·8	7·94	7·0 11·5	3	·714	15	AB
			103·3	10·71	C=14·8	3	·714	15	AC
h 1470	20 0·0	38 2	337·5	29·25	7·0 9·0	4	·741	14	
Espin 85	0·6	43 54	34·4	2·59	9·7 9·9	3	·655	15	
ρ 41	6·4	42 54	337·8	5·92	9·3 9·6	3	·730	15	
ρ 42	11·8	42 37	270·1	5·82	9·5 9·6	2	·723	15	
+43°·3627	26·8	43 23	231·2	7·01	9·1 9·1	2	·784	15	
+44°·3552	39·8	44 34	297·4	41·19	8·5 9·0	2	·863	14	
h 1571	39·8	41 9	55·1	7·04	9·6 9·9	3	·839	16	
h 1615	21 3·9	44 51	271·3	10·80	10·3 10·9	3	·746	14	
h 1618	6·4	43 35	132·7	2·41	12·0 12·5	2	·842	15	BC
			172·6	19·79	A= 8·0	3	·867	15	AB
Σ 2773	6·5	43 35	115·4	3·24	8·7 9·5	2	·796	15	AB
			64·8	22·23	C=13·0	3	·814	15	AC
			261·3	29·65	D=13·9	3	·814	15	AD
h 1634	14·8	42 19	137·2	26·88	8·8 9·0	3	·783	16	
h 1639	18·3	43 42	102·3	14·41	11·2 12·1	2	·744	14	
h 1640	18·8	43 43	43·0	9·32	9·5 10·7	3	·746	14	
Espin 99	25·5	44 33	199·9	5·11	8·5 11·5	2	·710	14	
Espin 100	25·7	44 47	157·9	4·73	8·6 9·0	3	·774	14	
Espin 683	54·6	44 13	249·0	8·75	8·5 9·2	2	·719	14	
h 1715	57·7	44 46	254·1	14·02	11·0 11·3	2	·744	14	
A 2497	22 13·4	42 2	196·4	5·14	9·0 12·0	2	·945	15	
Σ 2902	19·3	44 51	87·6	6·29	7·0 7·8	2	·702	14	
h 1765	21·4	42 46	187·6	14·71	9·4 9·7	2	·924	16	
A 415	46·6	43 33	331·4	4·07	8·7 10·2	3	·870	15	
h 1822	47·2	40 48	136·4	5·39	8·5 9·4	2	·995	16	
h 1831	51·2	42 31	91·3	14·79	9·7 10·1	2	·938	16	
A 788	23 19·7	45 13	270·0	4·20	8·8 13·7	1	·014	14	
h 1883	21·8	45 51	146·6	18·95	9·4 9·6	1	·014	14	
h 1932	59·2	42 2	302·2	7·01	8·4 8·6	2	·902	16	

Notes.

h 1978. *h* gives P 217°.
Espin 4. If the assumed P.M. is correct, this must be a physical pair.
Espin 13. Star C, added with the 24-inch.
Σ 599. Star C, added with the 24-inch.
h 2534. The measures give P.M. 0″·131 at 260°·8.
h 2559. *h*'s place is some 30 sec. too small in R.A.
h 2639. β's measure in 1906 should probably be 57″·23 for AC.
h 2869. A 14 mag. about the same distance N.
Espin 23. C was first seen in 1914.
h 1615. *h* reverses angle.
B.D. +43°·3627. A.G. (Bonn) double? Not in β's G.C.
h 1640. *h*'s angle 54°·0.
Σ 2902. Two distant *comites*, 13 mag., and a little pair in field at P 49° mags. 12·0, 12·8; P 79°·9; D 4″·97. 2 nts.
h 1831. *h*'s place 2′ too far N.

New Double Stars.

No.	B.D.	R.A. 1920. h m	Decl. ° ′	P. °	D. ″	Mags.	Nts.	1916.	
1480	+41° 4935	0 0·4	+42 3	90·9	6·33	9·1 12·5	4	·912	
1481	+43° 40	11·6	43 53	80·8	7·70	8·6 9·1	2	·017	
1482	+43° 41	12·2	43 45	187·9	3·76	9·5 14·0	2	·017	
1483	+42° 50	13·7	42 28	346·5	2·56	9·5 9·6	3	·923	AB
				14·7	2·27	12·0 12·1	2	·944	CD
				165·2	28·90		1	·925	AC
1484	+42° 52	13·9	42 56	341·8	2·35	9·5 12·7	2	·814	BC
				42·6	32·38	A= 8·5	2	·814	AB
1485	+42° 61	15·7	43 10	94·4	9·60	8·2 12·3	3	·844	
1486	+40° 77	19·9	40 48	350·7	6·06	8·8 12·0	3	·994	
1487	+42° 81	22·9	42 42	50·8	3·14	9·5 13·0	2	·861	
1488	+41° 133	41·9	41 41	280·4	6·78	8·6 8·9	2	·955	
1489	+40° 181	46·8	40 59	24·6	6·00	9·5 10·8	2	·999	
1490	+43° 180	51·0	44 3	144·1	2·75	9·0 12·0	3	·056	
1491	+43° 188	53·0	43 20	127·9	2·16	9·5 9·6	3	·035	
1492	+43° 198	56·0	43 23	75·3	3·04	9·1 11·5	2	·023	
1493	+43° 323	1 28·2	43 35	55·7	7·97	7·7 13·7	3	·014	
1494	+40° 323	29·4	40 52	5·0	5·99	9·4 9·7	2	·986	
1495	+42° 327	30·5	43 5	66·3	1·24	9·2 9·4	2	·814	
1496	+42° 333	31·7	43 10	191·0	7·09	8·5 12·0	2	·072	
1497	+43° 340	33·0	43 19	158·9	5·61	9·3 11·9	3	·079	
1498	+43° 369	41·9	43 23	359·1	3·50	9·5 9·6	2	·100	
1499	+42° 403	50·9	42 55	298·7	5·99	9·4 10·9	2	·825	
	+43° 440	2 5·4	43 39	181·6	10·20	8·8 10·0	2	·011	
1500	+42° 488	11·7	42 31	25·4	6·69	9·5 11·4	2	·891	
1501	+41° 450	16·8	41 46	95·7	6·56	8·6 13·2	2	·941	

No.	B.D.	R.A. 1920. h m	Decl. ° ′	P. °	D. ″	Mags.	Nts.	1916.	
1502	+40° 503	2 17·8	+41 10	162·7	6·70	9·4 10·8	2	·999	AB
				24·1	15·35	C= 9·5	2	·999	AC
1503	+41° 463	21·3	41 40	61·1	5·83	9·6 11·2	4	·974	
1504	+41° 491	29·9	41 29	100·0	5·95	9·2 13·5	2	·974	
1505	+42° 558	31·2	42 32	9·4	3·29	9·4 12·7	4	·682	
1506	+42° 601	35·9	42 20	353·4	5·71	8·4 14·0	2	·105	
1507	+41° 536	40·1	42 1	319·3	2·66	9·5 10·7	2	·941	
1508	+41° 566	49·2	41 51	57·7	3·43	9·5 9·7	2	·910	
1509	+41° 582	52·2	41 23	175·2	6·81	9·5 11·2	2	·958	
1510	+40° 637	53·0	40 46	324·2	4·97	9·3 9·3	2	·987	
1511	+42° 675	53·5	42 59	280·8	6·44	9·5 10·7	2	·072	
1512		56·0	41 13	116·9	3·75	9·5 9·7	3	·977	
1513	+41° 630	3 5·2	41 23	91·4	7·22	9·1 10·2	2	·999	
1514	+42° 739	12·7	42 44	48·7	1·19	9·5 9·6	2	·509	
1515	+41° 705	26·8	41 27	312·6	4·91	9·5 9·6	2	·974	
1516	+41° 713	29·2	41 54	87·1	4·20	9·3 11·0	2	·941	
1517	+42° 818	39·6	42 59	181·6	4·72	9·3 11·5	3	·397	
1518	+42° 825	41·7	42 19	240·0	5·95	9·4 11·2	2	·911	
1519	+42° 849	48·4	43 3	117·6	1·56	9·5 9·7	5	·060	
1520	+42° 888	59·7	43 0	4·4	6·10	9·5 9·6	2	·102	
1521	+42° 960	4 19·1	42 25	292·9	6·31	8·0 13·0	2	·958	
1522	+43° 967	19·6	43 15	311·0	8·89	7·5 13·7	2	·138	
1523	+43° 984	23·1	43 55	39·0	2·61	9·5 10·6	5	·072	
1524	+42° 1068	43·8	42 49	83·1	6·18	8·5 13·7	2	·903	
1525	+42° 1073	45·1	42 34	269·8	5·63	9·2 12·2	2	·987	
1526		45·4	42 51	103·3	1·93	9·8 10·0	3	·920	
1527	+43° 1137	50·1	43 12	138·3	2·87	9·5 10·7	3	·114	AB
				320·4	10·51	C= 13·0	3	·114	AC
1528	+44° 1211	5 21·5	44 12	45·3	5·61	9·4 9·5	2	·104	
1529	+44° 1229	25·7	44 27	236·3	2·59	9·3 13·7	3	·114	
1530	+43° 1346	39·5	43 20	14·5	5·19	9·1 9·1	2	·987	
1531	+43° 1382	49·0	43 52	301·5	4·03	9·3 9·4	2	·104	
1532	+43° 1482	6 5·5	43 40	231·8	7·61	9·5 12·0	3	·114	
1533	+43° 1499	9·1	43 42	254·3	6·47	8·5 12·5	2	·104	
1534	+43° 1531	18·4	43 26	92·1	1·78	9·4 9·7	2	·120	
1535	+42° 1595	37·6	42 37	74·2	6·95	8·6 13·2	2	·247	
1536	+41° 1615	7 6·0	41 46	11·8	1·42	9·1 9·4	2	·114	
1537	+41° 1718	38·4	40 55	354·9	6·73	9·3 13·2	2	·258	
1538	+42° 1794	53·7	42 16	216·6	2·53	9·2 10·0	2	·175	
1539	+41° 1766	54·0	41 14	100·8	1·32	9·2 9·4	3	·262	

No.	B.D.	R.A. 1920. h m	Decl. ° ′	P. °	D. ″	Mags.		Nts.	1916.	
1540	+40° 2014	8 9·3	+40 52	1·3	7·04	8·8	11·5	3	·269	
1541	+40° 2196	9 15·8	40 24	21·2	7·84	9·0	9·8	3	·303	
1542		10 18·6	40 35	196·7	3·52	9·5	9·6	2	·258	
1543	+40° 2370	42·2	40 30	12·1	3·20	9·5	12·5	2	·265	
1544	+40° 2642	13 14·0	40 25	267·6	6·97	8·1	10·0	2	·328	
1545	+40° 2643	14·5	40 25	88·9	6·34	9·5	13·5	2	·328	
1546	+40° 2648	16·8	40 16	221·9	2·87	9·2	13·5	2	·345	
1547		18·8	43 49	86·6	1·93	11·8	12·7	4	·420	BC
				327·2	27·53	A=	10·2	3	·408	AB
1548	+44° 2272	20·7	44 44	361·1	1·06	9·5	9·6	2	·387	
1549	+44° 2284	29·7	43 48	N.P.	5±	9·5	11	1	·427	
1550	+40° 2839	14 58·2	40 26	161·2	2·19	9·4	10·9	2	·387	
1551	+41° 2584	15 13·1	41 22	139·7	1·67	9·3	12·0	2	·417	
1552	+44° 2450	24·0	44 34	275·1	4·96	9·5	13·5	2	·372	
1553	+42° 2620	31·6	42 10	150·6	2·89	9·4	9·6	2	·454	
1554	+44° 2510	46·5	44 7	357·8	4·92	9·4	9·6	2	·372	
1555	+43° 2548	56·5	43 23	73·6	2·29	9·1	10·3	2	·407	
1556	+44° 2532	57·7	43 55	217·2	9·69	9·0	10·7	3	·374	
	+40° 3194	17 36·8	40 49	50·6	10·79	9·2	9·4	2	·588	
	+40° 3205	41·1	40 29	240·0	13·10	7·4	12·2	2	·618	
1557	+41° 2928	52·3	41 9	4·4	9·35	8·7	12·0	2	·602	
1558	+41° 2954	57·6	41 45	283·1	5·26	9·2	12·7	3	·587	
1559		18 26·4	41 54	181·2	1·85	9·8	10·5	3	·573	
1560	+41° 3144	44·6	41 53	340·9	7·95	9·5	11·6	2	·720	
1561	+41° 3147	44·8	41 15	122·2	6·97	9·2	10·5	2	·750	
1562	+42° 3309	19 16·1	42 6	323·7	5·75	8·5	8·9	2	·577	
1563	+42° 3446	42·4	42 15	206·9	7·57	9·2	10·8	2	·715	
1564	+42° 3476	45·4	42 26	200·9	7·91	9·2	10·6	2	·779	
1565	+42° 3481	45·8	42 22	230·2	3·41	11·5	12·3	4	·793	BC
				36·2	38·23	A=	9·2	3	·790	AB
	+42° 3514	50·9	42 18	340·3	9·51	8·7	9·3	2	·834	
1566	+42° 3630	20 8·0	42 14	350·2	5·60	9·5	9·6	2	·715	
1567	+42° 3656	12·4	42 15	177·4	2·02	9·3	9·5	2	636	
1568	+41° 3685	13·3	41 39	186·7	2·62	12·2	12·3	4	·894	BC
				299 0	11·45	A=	9·5	2	·899	A
1569		29·8	42 15	268·6	2·68	9·3	10·6	3	·697	
1570	+41° 3852	38·1	41 41	119·1	4·68	9·5	9·6	2	·795	
1571	+41° 3862	39·2	41 10	36·0	6·41	9·5	9·6	3	·810	
1572	+41° 3870	40·6	41 43	166·5	5·99	9·2	13·5	2	·779	
1573	+41° 3890	44·2	41 42	74·1	4·52	9·0	12·7	3	·839	

No.	B.D.	R.A. 1920. h m	Decl. ° ′	P. °	D. ″	Mags.		Nts.	1916.	
1574	+41° 3922	20 49·7	+41 17	155·6	1·64	9·6	9·7	3	·936	
1575	+42° 3924	54·8	42 38	357·6	5·39	9·5	13·7	2	·632	
1576	+42° 3930	55·6	42 32	6·7	4·49	9·5	14·0	3	·647	
1577	+41° 3967	58·5	42 10	38·9	6·12	9·3	9·8	2	·670	
1578	+42° 3948	58·7	42 12	228·9	2·29	9·3	9·8	2	·670	
1579	+42° 3961	21 1·8	42 32	269·8	3·52	9·4	12·0	2	·728	
1580	+42° 3974	3·2	42 32	88·1	3·31	9·5	9·6	2	·728	
1581	+41° 4047	10·6	41 54	49·2	4·66	9·5	9·7	2	·961	
1582	+41° 4063	12·3	41 59	133·4	3·89	9·2	10·5	2	·899	
	+42° 4055	15·6	42 28	306·7	13·31	9·3	10·0	3	·697	
1583	+41° 4093	15·9	42 3	76·1	5·89	9·6	10·2	2	·851	BC
				239·6	14·28	A= 9·4		2	·851	AB
1584	+41° 4096	16·4	41 42	113·3	3·43	9·5	10·8	2	·913	
1585	+42° 4065	17·6	42 35	265·8	4·97	9·3	12·5	3	·762	
1586	+40° 4721	22 1·3	41 2	39·2	4·37	9·5	9·6	3	·866	
1587		14·7	40 37	299·7	3·91	9·5	9·6	2	·961	
1588	+40° 4781	15·1	40 40	96·0	5·89	9·4	9·5	2	·961	
1589	+41° 4464	17·0	41 57	178·3	9·30	8·6	9·7	3	·796	
1590	+41° 4472	18·1	41 23	167·0	7·95	9·5	10·7	3	·895	
1591	+42° 4382	19·9	42 38	160·1	2·31	9·4	10·0	2	·672	
1592	+41° 4536	28·5	41 32	312·0	7·25	9·1	11·2	2	·787	
1593	+40° 4868	34·0	41 13	145·4	6·73	9·3	12·7	2	·899	
1594	+41° 4650	53·5	41 24	94·8	8·97	8·7	9·7	2	·891	
1595	+40° 4972	58·0	41 11	337·9	2·24	9·4	9·5	2	·961	
1596	+41° 4680	23 2·0	42 7	76·4	5·70	11·7	12·7	3	·789	BC
				21·9	35·77	A= 7·6		2	·787	AB
1597	+41° 4802	26·5	41 31	90·0	7·70	8·4	12·5	2	·814	
1598		30·8	40 25	273·5	2·76	9·4	9·5	2	·981	
1599	+40° 5125	34·5	40 20	347·0	3·81	9·3	9·4	2	·987	
1600	+41° 4835	34·8	42 14	301·8	6·03	9·5	9·7	3	·021	

Notes.

1522. Also a 10 mag., P 102°·9, D 31″·5.
1526. Forms a distant *comes* to B.D. +42°·1079 at P 237°.
1531. There is a 9·3 mag. at P 292°·6, D 41″, and four other stars in the group.
1542. P of two stars, S of B.D. +40°·2322.
1557. An 11 mag. at P 37°·8, D 28″, a more distant 13 mag. at P 309°.
1559. Forms a distant *comes* to B.D. +41°, 3063 at 87°·2.
1562. A 12 mag. at P 56°·3, D 30″.
1566. On the second night, the N star seemed to be slightly the brighter.
1568. Middle of three stars on N border of a nebula. R.A. about 20 sec. less than in Argelander.
69. This agrees in R.A. with B.D. +42°, 3780, but is 4′ S in declination. There is no star in Argelander's place.
1575. SF of two.
1576. N star of a pair in Argelander.
1589. Also a 12 mag. at P 240°, D 22″.

On the Question of Extinction of Light in Space and the Relations between Stellar Magnitudes, Distances, and Proper Motions. By Dr J. Halm, Ph.D. (Plates 1, 2.)

The standardisation of the photographic magnitudes of stars near the South Pole, and subsequent differential comparisons between the south-polar sequence and stars in selected areas of the zone 45° southern declination, have established a sufficiently reliable framework of standard photographic magnitudes for an examination of the magnitudes of the C.P.D. catalogues with the view of ascertaining their relation to a fundamental photographic system. A detailed account of this work will be published in the *Annals of the Cape Observatory.* The present communication is limited to the announcement and discussion of some results whose bearing on questions of the structure of the universe may be of general interest.

§ 1. *The Statistical Relation between Magnitudes and Star Counts.*

The work of standardising the magnitudes of the C.P.D. was primarily undertaken with the view of testing the accuracy of its photometric basis through a comparison with results derived from stars of the northern hemisphere. As the medium of comparison the relation between magnitudes and star counts was utilised. With regard to the southern hemisphere, the records of the star numbers in the C.P.D. catalogues discussed by Stratonoff in his "Etudes sur la structure de l'univers," *Publications de l'Observatoire astronomique et physique de Tachkent*, No. 3, form a suitable basis, while the extensive work of Chapman and Melotte (*Memoirs R.A.S.*, vol. lx., part iv.), based chiefly on the Greenwich Astrographic Catalogues and the northern portion of the Franklin-Adams charts, establishes the relation between photographic magnitudes and star numbers for the northern portion of the heavens.

The first question was to examine how far the relationship between star numbers and magnitudes agrees between the two hemispheres.

In Appendix II. of his publication Stratonoff gives the numbers of C.P.D. stars per square degree in groups comprised between the C.P.D. magnitudes 1 to 6·0; 6·1 to 6·5; 6·6 to 7·0; 7·1 to 7·5; 7·6 to 8·0; 8·1 to 8·5; 8·6 to 9·0; 9·1 to 9·5 and 9·6 >.

These "star densities" are quoted for contiguous areas whose positions are recorded in Appendix I. of Publication No. 2. The material utilised in the present investigation is comprised within the zone between − 20° and − 50° declination, containing 432 areas numbered from 1189 to 1620. After reducing the C.P.D. magnitudes to the Cape standard system, the numbers of stars,

N,* brighter than certain specified magnitudes were found as shown in Table I.

TABLE I.

Limiting Magnitudes.		log N (Whole Sky).
C.P.D.	Stand. Ptg.	
6·5	6·15	3·565
7·0	6·83	3·910
7·5	7·57	4·265
8·0	8·36	4·658
8·5	9·20	4·984
9·0	10·04	5·359
9·5	10·85	5·653
10·15	11·87	6·105

In comparing these results with those derived by Chapman and Melotte, it should be noted that their magnitudes are mainly based on the Harvard system of the north-polar sequence published in H.C. 170. Professor Seares has pointed out that the scale of this system differs sensibly from that derived by fundamental observations made at Mount Wilson. From the figures published in table iii. of his paper on "A Comparison of the Harvard and Mount Wilson Scales of Photographic Magnitude" (*Astrophysical Journal*, vol. xli., 1915), the differences (Mount Wilson—H.C. 170) in order of magnitude are shown in Table II.

TABLE II.

m.	(Mt. W.–H.).	*m*.	(Mt. W.–H.).
6·0	−0·02	12·0	+0·31
7·0	+0·02	13·0	+0·32
8·0	+0·09	14·0	+0·33
9·0	+0·18	15·0	+0·31
10·0	+0·25	16·0	+0·25
11·0	+0·29	17·0	+0·17

According to Seares, the results of the Göttingen and Yerkes actinometry support the Mount Wilson results.

In Table III. are shown for specified values of *m* (standard photographic)—

(*a*) the values of log N derived from the C.P.D. star counts (Stratonoff) and interpolated graphically from the observed values of Table I.;

(*b*) the values of log N derived by Chapman and Melotte and based on the photographic system of H.C. 170;

(*c*) the values of log N derived from the same material as under (*b*), but based on the Mount Wilson photographic system.

* The numbers N are related to the numbers B counted in the area ω (1 square degree) by

$$N = \frac{4\pi}{\omega} \cdot B,$$

$$\log \frac{\pi}{\omega} = 4{\cdot}616.$$

TABLE III.

m.	log N (C.P.D.).	log N (C. & M., H.C. 170).	log N (C. & M., Mt. Wilson).
4·0	...	(2·48)	...
5·0	...	(2·99)	(3·02)
6·0	3·50	3·50	3·51
7·0	4·00	3·99	3·98
8·0	4·47	4·51	4·47
9·0	4·92	4·99	4·92
10·0	5·33	5·43	5·33
11·0	5·73	5·84	5·73
12·0	6·12	6·22	6·12
13·0	...	6·57	6·47
14·0	...	6·88	6·79
15·0	...	7·19	7·09
16·0	...	7·47	7·39
17·0	...	7·74	7·68

The figures of this table indicate that the Cape photographic system is in agreement with the Mount Wilson system both as regards scale and zero-point. This fact lends support to the correctness of the Mount Wilson scale, in addition to the evidence in its favour shown by the Göttingen and Yerkes observations. The subsequent discussion will therefore be based on the values of log N given in the second and fourth columns of Table III.

Next it seemed important to examine the relation between m and log N with reference to the galaxy. Two regions were selected, the one comprising stars in close proximity to the Milky Way (galactic latitude, $\beta = \pm 2^\circ{\cdot}5$), and the other containing the stars near the southern galactic pole ($\beta = -74^\circ$). The resulting figures are shown in Table IV., first and second column. The third column of the same table shows the values of log N_0 taken from Table III. for the corresponding magnitudes, the fourth column giving the logarithm of the ratio N/N_0.

TABLE IV.

Galaxy: $\beta = \pm 2^\circ{\cdot}5$.				Galactic Pole: $\beta = -74^\circ$.			
m (Ptg.).	log N.	log N_0.	log N/N_0.	m (Ptg.).	log N.	log N_0.	log N/N_0.
6·14	4·02	3·57	+0·45	7·58	3·85	4·28	−0·43
7·00	4·40	4·00	+0·40	8·28	4·18	4·60	−0·42
7·85	4·86	4·40	+0·46	9·01	4·53	4·93	−0·40
8·71	5·24	4·80	+0·44	9·75	4·80	5·24	−0·44
9·70	5·68	5·22	+0·46	10·42	5·08	5·51	−0·43
10·64	6·04	5·60	+0·44	11·06	5·27	5·76	−0·49
11·85	6·54	6·07	+0·47	11·86	5·57	6·07	−0·50

The ratios N/N_0 show a remarkable constancy throughout the available range of magnitudes, in agreement with the result derived by Pickering from the visual magnitudes, and also with the general conclusions arrived at by Chapman and Melotte in their discussion of the photographic magnitudes of the northern hemisphere. The ratio (Galaxy : Pole) is, however, much in excess of that found by Chapman and Melotte, viz. 7·8 as against 3·1. The difference is doubtless to a large extent real, and due to an apparently greater contrast in star wealth. It appears also to be borne out by the Johannesburg Franklin-Adams plates (table xvi. of Chapman and Melotte's paper). If, within the zones centred on $-30°$, $-45°$, and $-60°$ declination, the numbers of stars down to the 17th magnitude are compared between regions $\beta \leqq 5°$ and $\beta > 70°$, the ratio 6·3 is found, *i.e.* double the amount found from the northern stars. It is quite likely, however, that the ratio derived from Table IV. may be somewhat exaggerated by seasonal changes in the zero-point of the magnitude scale, the probability of such a defect being suggested by the fact that the regions near the galactic pole were always observed at the same (hottest) time of the year.

Attention was next directed to a comparison of the standardised photographic magnitudes with a visual system. Professor Kapteyn has kindly placed at our disposal a detailed comparison between the C.P.D. magnitudes and the visual magnitudes of the Harvard system for the zone here examined. Through the medium of the C.P.D. we may thus compare the visual and photographic systems. Six regions, according to galactic position, were examined. In each region the stars were collected in groups arranged in order of C.P.D. magnitude. The third column of Table V. shows the differences (Photographic-Visual) derived from the general mean of all the regions, while columns 4–9 exhibit the departures from these mean differences for each region and magnitude.

Table V.

$m.$ (C.P.D.)	$m.$ (Photogr.)	(Ptg.-Vis.)	$\beta=67°9.$	$55°4.$	$38°9.$	$20°3.$	$13°1.$	$5°1.$
7·5	7·60	+0·56	+0·32	+0·23	+0·17	−0·11	−0·26	−0·35
8·0	8·38	+0·55	+0·37	+0·21	+0·17	−0·13	−0·26	−0·37
8·5	9·15	+0·55	+0·41	+0·27	+0·11	−0·10	−0·29	−0·38
9·0	10·00	+0·67	+0·43	+0·33	+0·10	−0·16	−0·29	−0·40
9·5	10·85	+0·86	+0·40	+0·34	+0·03	−0·11	−0·29	−0·38
10·0	11·59	+0·89	+0·37	+0·29	+0·04	−0·11	−0·27	−0·30
		Mean :	+0·38	+0·28	+0·10	−0·12	−0·28	−0·36

The figures of this table point to several interesting facts. First, it appears from the third column that the difference (Ptg.-Vis.) increases with increasing magnitude, *i.e.* fainter stars are somewhat redder than bright stars. Secondly, the constancy of the figures shown in the vertical columns 4–9 indicates that the rate at which

fainter stars become redder is the same for all regions. Thirdly, the means at the bottom of these columns show that the stars of al magnitudes become increasingly *bluer* as the galaxy is approached.

§ 2. *The Physical Relation between Magnitudes and Star Numbers.*

Having established in the figures of Table III. the statistical relation between m and N on what may be considered a consistent photometric basis, an attempt was made to examine this relation from the physical point of view.

It is well known that in a universe in which the stars are uniformly distributed, and in which the star's light is not impeded by space absorption, the relation between apparent magnitudes and star numbers is expressed by the equation:

$$m = \tfrac{5}{3} \log \mathrm{N} + \text{constant} \quad . \quad . \quad . \quad (1)$$

This relation holds whatever may be the law according to which the absolute luminosities are distributed, provided that the luminosity law is independent of the distance.

It is quite evident from the figures of Table III. that equation (1) does not represent the observed relation between m and N. Opinions as to the possible cause of the observed differences are divided, the majority of astronomers being inclined to attribute them to a gradual thinning out of star matter with increasing distances from us. The alternative assumption of extinction of the star's light during its passage through space, though its possibility is universally admitted, has hitherto received little support. Perhaps the only important attempt to reconcile observation with this point of view is that of Comstock in his paper in No. 558 of the *Astronomical Journal*. The criticism of Comstock's views by Kapteyn in No. 566 of the same publication centres on the fact that extinction of the amount suggested by Comstock would lead to an enormous and obviously quite inadmissible increase of the star density with distance. While the force of his arguments against the amount of extinction must be admitted, the trend of his reasoning is, however, not directed against the assumption itself. In fact, some of his calculations appear to bear out a coefficient of extinction about one-third of that derived by Comstock.

A significant light appeared to be thrown upon the question by the following attempt at representing the observed relation between m and N.

Instead of equation (1) the modified form

$$m = \tfrac{5}{3} \log \mathrm{N} + \alpha/\bar{\pi} + \text{constant} \quad . \quad . \quad . \quad (2)$$

was adopted, where $\bar{\pi}$ represents the mean parallax of stars of magnitude m.

The relation between magnitude and mean parallax being

known through Kapteyn's and Comstock's investigations of parallactic motions for stars brighter than 12th magnitude, we are enabled to test the applicability of equation (2), at least within this range.

Kapteyn's general results are given on page 12 of the Groningen Publications, No. 8, the derived parallaxes being based on the value 16·7 km./sec. for the solar motion. They extend over a range of magnitudes from 2·7 to 8·9. A further value of $\bar{\pi}$ for fainter stars of average magnitude 10·5 may be added to Kapteyn's series through an investigation of Comstock (*Astron. Journ.*, No. 558). Reducing Kapteyn's parallaxes to Campbell's value of solar motion, 19·5 km./sec., the values of $\bar{\pi}$ in the second column of Table VI. were obtained corresponding with the magnitudes of the first column. The third column of the same table shows the values of log N related to the same magnitudes. They were interpolated from the column headed "Harvard" in table xiii. of Chapman and Melotte's paper. Since the magnitudes to which the parallaxes refer, except the last, which is somewhat uncertain, are based on the Potsdam system, a small constant correction for reduction to Harvard should have been added to the values of log N. This correction, however, is irrelevant for our purpose, and has therefore been omitted.

TABLE VI.

m.	$\bar{\pi}$.	log N.
2·73	0·0362	2·00
4·14	0·0194	2·73
5·07	0·0139	3·21
6·01	0·0122	3·68
6·93	0·0085	4·13
8·89	0·0060	5·08
10·5	0·0046	5·77

In Table VIA. the observed magnitudes are compared with values computed from the equation:

$$m = \tfrac{5}{3} \log N + 0{\cdot}00764/\bar{\pi} - 0{\cdot}82.$$

TABLE VIA.

m (Obs.).	m (Comp.).	O−C.
2·73	2·72	+ ·01
4·14	4·12	+ ·02
5·07	5·08	− ·01
6·01	5·94	+ ·07
6·93	6·96	− ·03
8·89	8·92	− ·03
10·5	10·47	+ ·03

The agreement is surprisingly close, and suggests a physical interpretation of equation (2), viz. that the observed facts may perhaps be explained on the assumption of a universe in which the stars are uniformly distributed, their light suffering a minute amount of extinction during its passage through space.

The question then arises whether such an assumption would be sufficient to explain the observed relationship between m and N down to the faintest stars included in Chapman and Melotte's investigation.

The evidence afforded by Table VII. must, however, not be taken to indicate uniformity of star density throughout the whole of the universe. In fact, the pronounced crowding of stars near the galaxy emphasises the futility of an attempt to explain the structure of the universe on the basis of such a simple assumption. It must be considered, however, that in summing up the stars over the whole sky, those contributed by the regions near the galaxy must necessarily preponderate. Consequently, the resulting relation between star numbers and magnitudes may be considered typical of *low* galactic latitudes, but cannot be taken without further proof to represent the conditions prevailing near the galactic poles.

The course adopted in the subsequent investigation is, first, to examine how far the combined assumptions of uniform density and of extinction of light in space account for the statistical facts relating to the stars as a whole, and afterwards to compare the theoretical conclusions with the conditions prevailing in special areas selected with reference to the galaxy.

§ 3. *Relations between Magnitudes, Star Numbers, and Distances on the Assumptions of Uniform Density and Extinction of Light.*

The assumption of *uniform* density leads to a simple analytical aspect of the problem. Consider the stars enclosed in a spherical shell of thickness dr of which the sun is the centre. The volume of the shell is

$$4\pi r^2 dr.$$

Let $\nu(r)$ represent the number of stars contained in unit volume at distance r, then the total number of stars contained between the radii $(r \pm \frac{1}{2}dr)$ is

$$4\pi\nu(r)r^2 dr.$$

Let further the proportion of stars whose absolute magnitudes lie between $(\mathrm{M} \pm \frac{1}{2}d\mathrm{M})$ be represented by

$$\phi(\mathrm{M})d\mathrm{M}.$$

Consequently the number of stars whose distances lie between $(r \pm \frac{1}{2}dr)$ and whose absolute magnitudes are between $(\mathrm{M} \pm \frac{1}{2}d\mathrm{M})$ is represented by

$$4\pi\nu(r)r^2 dr\phi(\mathrm{M})d\mathrm{M} \quad . \quad . \quad . \quad . \quad (3)$$

We may further introduce the apparent magnitudes, m, by the relation:

$$M = m - 5 \log r - ar \quad . \quad . \quad . \quad (4)$$

where ar represents the effect of extinction in space assumed proportional to the distance.

We may replace dM in (3) by its equivalent form in terms of m through differentiating (4), considering r constant. Hence

$$dM = dm,$$

so that (3) expressed in terms of m and r takes the form:

$$4\pi\nu(r)r^2 dr\phi(m - 5 \log r - ar)dm \quad . \quad . \quad (5)$$

and represents therefore the number of stars whose distances lie between $(r \pm \frac{1}{2}dr)$ and whose apparent magnitudes are between $(m \pm \frac{1}{2}dm)$.

So far the density $\nu(r)$ has been treated as an unknown function of the distance. We shall now assume that

$$\nu(r) = \nu_0 = \text{constant},$$

and hence write instead of (5):

$$4\pi\nu_0 r^2 dr\phi(m - 5 \log r - ar)dm \quad . \quad . \quad . \quad (6)$$

Further, writing:

$$r_1 = r - \tfrac{1}{2}dr\,;\; r_2 = r + \tfrac{1}{2}dr,$$

(6) may be represented in the form:

$$\tfrac{4}{3}\pi\nu_0(r_2^3 - r_1^3)\phi(m - 5 \log r - ar)dm \quad . \quad . \quad (7)$$

where obviously $\frac{4}{3}\pi\nu_0$ is the total number of stars contained in a sphere of radius 1 (1 parsec).

With regard to the function $\phi(M)$, its analytical form seems to be well established by the investigations of Seeliger, Kapteyn, and Hertzsprung. These researches agree in assigning to $\phi(M)$ the form:

$$\frac{h}{\sqrt{\pi}} e^{-h^2(M - M_0)^2},$$

an expression indicating that the absolute magnitudes are distributed about a mean value M_0 in accordance with the law of accidental errors. This analytical expression for $\phi(M)$ will therefore be adopted, and hence we shall finally write (7) in the form:

$$\left.\begin{aligned} N_0(r_2^3 - r_1^3)\frac{h}{\sqrt{\pi}} e^{-h^2[m - 5 \log r - ar - M_0]^2}dm \\ N_0 = \tfrac{4}{3}\pi\nu_0. \end{aligned}\right\} \quad . \quad (8)$$

This expression enables us to compute the number of stars whose apparent magnitudes lie between any specified limits $(m \pm \frac{1}{2}dm)$, and whose distances are between any specified and sufficiently

narrow limits r_1, r_2. The four unknowns N_0, h, a, and M_0 may be ascertained so as best to satisfy the data of Table III.

After a number of somewhat laborious trials the following constants were selected:

$$N_0 = 0{\cdot}3076\,;\ h = 0{\cdot}249\,;\ a = 0{\cdot}00305\,;\ M_0 = 3{\cdot}95 \quad . \quad (9)$$

The scheme according to which the star numbers were computed was adopted in accordance with that used by Kapteyn. The successive steps in distance were so chosen that the interval between two successive boundaries was determined by the condition:

$$\log r_2 - \log r_1 = 0{\cdot}200.$$

As the mean radius, r, within each of these shells the value

$$r = \tfrac{3}{4}\frac{r_2^4 - r_1^4}{r_2^3 - r_1^3}$$

was adopted. This radius obviously represents the mean radius for *all* the stars contained within the shell. Table VII. shows the distances and corresponding values of ar used in the computations, as well as the quantities:

$$A = N_0(r_2^3 - r_1^3)\frac{h}{\sqrt{\pi}}.$$

TABLE VII.

$\log r_2$.	$\log r_1$.	$\log r$.	ar.	log A.
3·64	3·44	3·566	11·22	9·430
3·44	3·24	3·366	7·08	8·830
3·24	3·04	3·166	4·47	8·230
3·04	2·84	2·966	2·82	7·630
2·84	2·64	2·766	1·78	7·030
2·64	2·44	2·566	1·12	6·430
2·44	2·24	2·366	0·71	5·830
2·24	2·04	2·166	0·45	5·230
2·04	1·84	1·966	0·28	4·630
1·84	1·64	1·766	0·18	4·030
1·64	1·44	1·566	0·11	3·430
1·44	1·24	1·366	0·07	2·830
1·24	1·04	1·166	0·05	2·230
1·04	0·84	0·966	0·03	1·630
0·84	0·64	0·766	0·02	1·030
0·64	0·44	0·566	0·01	0·430

With the data of Table VII. the number of stars of apparent magnitudes lying between $(m \pm \frac{1}{2})$ have been computed by means of (8) and (9).

The resulting figures, computed with three-place logarithms, are shown in Table VIII. for the magnitudes from 1·5 to 17·5.

TABLE VIII.

Apparent magnitude $= m$ (phot.).

log r_2.	log r_1.	1·5-2·5.	2·5-3·5.	3·5-4·5.	4·5-5·5.	5·5-6·5.	6·5-7·5.	7·5-8·5.	8·5-9·5.	9·5-10·5.	10·5-11·5.	11·5-12·5.	12·5-13·5.	13·5-14·5.	14·5-15·5.	15·5-16·5.	16·5-17·5.
3·64	3·44													1	5	45	347
3·44	3·24									2	15	114	766	4560	23880	110100	451900
3·24	3·04						2	13	94	582	3200	15490	66680	252900	847200	2500000	6532000
3·04	2·84				2	12	78	449	2270	10160	40360	140900	436500	1188000	2871000	6110000	11480000
2·84	2·64	0·1	0·4	3	21	121	608	2710	10660	37150	114300	310400	744700	1577000	2952000	4887000	7145000
2·64	2·44	0·4	2·8	17	89	416	1710	6208	19860	56360	140900	311100	608200	1047000	1600000	2153000	2559000
2·44	2·24	1·5	8·6	43	189	740	2547	7762	20900	49660	104500	194100	317600	460200	588800	665300	665300
2·24	2·04	3·3	16·0	69	259	865	2553	6654	15310	31190	56100	88920	124800	154500	169400	163700	140000
2·04	1·84	5·2	21·8	81	264	762	1941	4375	8730	15350	23870	32810	39810	42660	40360	33810	24950
1·84	1·64	6·2	22·9	74	211	532	1185	2333	4055	6223	8452	10120	10720	10030	8280	6487	3899
1·64	1·44	6·3	20·1	57	142	313	611	1054	1603	2158	2565	2691	2495	2040	1479	949	533
1·44	1·24	5·3	14·8	37	81	157	270	408	547	647	675	622	508	366	233	131	65
1·24	1·04	3·8	9·4	21	40	69	104	138	163	170	156	127	91	58	32	16	7
1·04	0·84	2·4	5·3	10	17	26	35	41	43	39	32	23	14	8	4	2	1
0·84	0·64	1·3	2·6	4	7	9	10	11	10	8	6	4	2	1			
0·64	0·44	0·8	1·1	2	2	3	3	3	2	1	1	1					
0·44	0·24	0·3	0·4	1	1	1	1	1									
0·24	0·04	0·1	0·1														

From these figures the following quantities may be calculated :—

(*a*) The number of stars, N, whose magnitude is brighter than *m*.

(*b*) The mean parallax of stars of magnitude *m*.

(*c*) The mean distance of stars of magnitude *m*.

(*d*) The mean absolute magnitude, $\bar{M}$, of stars of magnitude *m*.

These four quantities, with additional data which will be discussed further on, are collected in Table IX.

TABLE IX.

m.	log N.	log $\bar{\pi}$.	log $\bar{r}$.	$\bar{M}$.	log $(\bar{\pi}\, N^{0\cdot234})$.	log $(\bar{r} N^{-0\cdot183})$.	$\frac{\Delta \log N}{\Delta \log \bar{\pi}}$	$\frac{\Delta \log N}{\Delta \log \bar{r}}$
1·50	1·16	−1·237	1·737	−6·16	−0·97	1·52		
2·50	1·708	−1·395	1·856	−5·99	−0·995	1·543	3·52	4·78
3·50	2·248	−1·546	1·965	−5·69	−1·020	1·554	3·66	5·06
4·50	2·775	−1·686	2·067	−5·35	−1·037	1·559	3·77	5·23
5·50	3·283	−1·817	2·163	−4·99	−1·049	1·562	3·95	5·38
6·50	3·774	−1·939	2·253	−4·61	−1·056	1·562	4·07	5·53
7·50	4·245	−2·053	2·337	−4·22	−1·060	1·560	4·17	5·63
8·50	4·697	−2·160	2·417	−3·82	−1·061	1·558	4·26	5·69
9·50	5·127	−2·260	2·492	−3·41	−1·060	1·554	4·32	5·79
10·50	5·536	−2·354	2·562	−2·98	−1·059	1·549	4·36	5·78
11·50	5·924	−2·443	2·630	−2·55	−1·057	1·547	4·38	5·67
12·50	6·289	−2·526	2·696	−2·13	−1·055	1·545	4·33	5·54
13·50	6·633	−2·607	2·759	−1·71	−1·055	1·546	4·28	5·42
14·50	6·956	−2·682	2·819	−1·31	−1·055	1·546	4·29	5·26
15·50	7·259	−2·753	2·878	−0·92	−1·054	1·550	4·22	5·08
16·50	7·541	−2·822	2·934	−0·54	−1·057	1·554	4·08	4·95
17·50	7·804	−2·888	2·988	−0·19	−1·062	1·560	3·96	4·82
18·50	8·052	−2·951	3·040	+0·15	−1·067	1·567	3·86	4·61
19·50	8·279	−3·011	3·091	+0·45	−1·074	1·576	3·75	4·38
20·50	8·491	−3·068	3·141	+0·73	−1·081	1·587		

We proceed to discuss the results in succession by comparison with observation.

With regard to the relation between *m* and N, Table X. shows the comparison between the theoretical values and the results of observation given in Table III.

TABLE X.

m.	log N (Obs.).	log N (Theory).	Diff.	*m*.	log N (Obs.).	log N (Theory).	Diff.
5·0	3·02	3·029	−·009	12·0	6·12	6·107	+·013
6·0	3·51	3·528	−·018	13·0	6·47	6·461	+·009
7·0	3·99	4·009	−·019	14·0	6·79	6·794	−·004
8·0	4·47	4·471	−·001	15·0	7·09	7·107	−·017
9·0	4·92	4·912	+·008	16·0	7·39	7·400	−·010
10·0	5·33	5·332	−·002	17·0	7·68	7·673	+·007
11·0	5·73	5·730	·000				

The agreement between observation and theory may be considered highly satisfactory, even to the faintest stars so far included in the statistical investigation of Chapman and Melotte.

The constants of (9) have been adapted to represent the relation between N and the *photographic* magnitudes. If this relation refers to *visual* magnitudes and if the difference (Photg. – Visual) is known, then by converting the visual into photographic magnitudes the observed relation should be in agreement with that of Table IX. Or *vice versa*, if the magnitude corresponding with the observed log N is taken from this table, the difference between the tabular and observed magnitude should be identical with the colour index (Photg. – Visual).

Chapman and Melotte in their table xiii. show the relation between the visual magnitudes and N according to Harvard. The figures quoted there are given in the first two columns of Table XI. On the other hand, from Table IX. we find the photographic magnitudes of the third column, and also the logarithms of the mean parallaxes.

TABLE XI.

m (Visual).	log N.	m (Phot.).	log $\bar{\pi}$.	(Phot. – Visual.)
2·0	1·61	2·32	– 1·37	+0·32
3·0	2·14	3·30	– 1·52	+0·30
4·0	2·66	4·28	– 1·66	+0·28
5·0	3·17	5·28	– 1·79	+0·28
6·0	3·68	6·32	– 1·92	+0·32
7·0	4·17	7·34	– 2·04	+0·34
8·0	4·66	8·42	– 2·15	+0·42
9·0	5·13	9·50	– 2·26	+0·50
10·0	5·57	10·58	– 2·36	+0·58
11·0	5·98	11·65	– 2·45	+0·65
12·0	6·35	12·67	– 2·54	+0·67
13·0	6·68	13·64	– 2·62	+0·64

The last column shows a slight increase of the colour index with increasing magnitudes, of the same amount as that indicated by the photometric results in the third column of Table V.

It is readily shown that the relation between magnitudes and mean parallaxes is likewise correctly represented. Fig. 1 presents this relation in graphical form, the abscissæ being represented by the apparent visual magnitudes and the ordinates by the values of log $\bar{\pi}$. The observed values (Kapteyn) derived from the parallactic motions (Table VI.) are indicated by small circles. It will be noticed that the curve does not deviate much from a straight line, thus confirming the approximate correctness of the empirical law of Kapteyn within the range of these observations. Reduced to

Campbell's solar motion, the numerical relation between magnitude and mean parallax, according to Kapteyn, is

$$\log \bar{\pi} = -1{\cdot}869 - 0{\cdot}108(m - 5{\cdot}5) \quad . \quad . \quad (10)$$

This equation is represented by the straight line of fig. 1. The

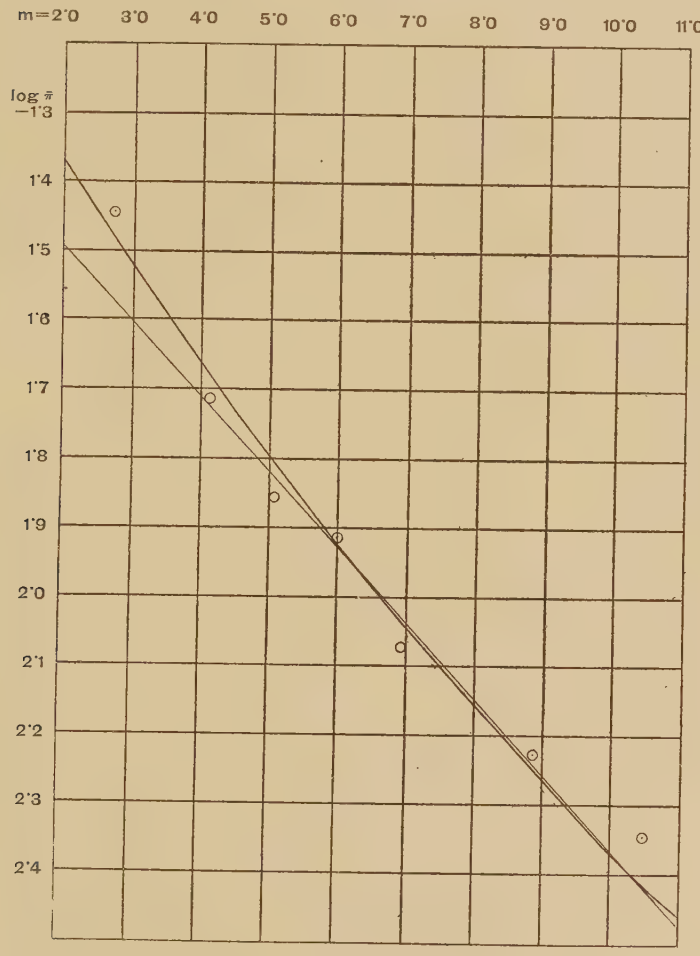


Fig. 1.

theoretical curve adapts itself well to the slope of this line in the region of the fainter stars, but deviates appreciably from it for the brighter stars, following fairly closely the curve derived from observation.

The agreement between the theoretical curve and Kapteyn's equation is specially interesting on account of recent criticism directed against the magnitude coefficient of the latter, which, it

is maintained, is too large. This point will be dealt with in a subsequent section (§ 5). It will be pointed out that determinations of this coefficient based on the *motus peculiares* must differ from those based on parallactic motions on account of the fact that the peculiar motions depend in some degree on the absolute magnitudes.

We may further compare the conditions of a universe in which light undergoes extinction during its passage with those of a universe in which $\alpha = 0$, but the assumed laws of density and luminosity are the same. Table XII. contains the values of log N, log $\bar{\pi}$, and $\bar{M}$ for the two assumptions $\alpha = 0$ and $\alpha = 0{\cdot}00305$.

TABLE XII.

m (Ptg.).	$\alpha = 0$.			$\alpha = 0{\cdot}00305$.		
	log N.	log $\bar{\pi}$.	$\bar{M}$.	log N.	log $\bar{\pi}$.	$\bar{M}$.
1·5	1·28	−1·37	−7·15	1·16	−1·24	−6·1
2·5	1·88	−1·57	−7·15	1·71	−1·40	−6·0
3·5	2·48	−1·77	−7·15	2·25	−1·55	−5·7
4·5	3·08	−1·97	−7·15	2·78	−1·69	−5·3
5·5	3·68	−2·17	−7·15	3·28	−1·82	−5·0
6·5	4·28	−2·37	−7·15	3·77	−1·94	−4·6
7·5	4·88	−2·57	−7·15	4·24	−2·05	−4·2
8·5	5·48	−2·77	−7·15	4·70	−2·16	−3·8
9·5	6·08	−2·97	−7·15	5·13	−2·26	−3·4
10·5	6·68	−3·17	−7·15	5·54	−2·35	−3·0
11·5	7·28	−3·37	−7·15	5·92	−2·44	−2·5
12·5	7·88	−3·57	−7·15	6·29	−2·53	−2·1
13·5	8·48	−3·77	−7·15	6·63	−2·61	−1·7
14·5	9·08	−3·97	−7·15	6·96	−2·68	−1·3
15·5	9·68	−4·17	−7·15	7·26	−2·75	−0·9
16·5	10·28	−4·37	−7·15	7·54	−2·82	−0·5

According to these figures, the presence of extinction causes a considerable diminution in the number of stars brighter than any specified magnitude. The comparison of the mean parallaxes $\bar{\pi}$ for the same magnitudes also reveals an enormous reduction in the extent of our vision.

Finally, the table shows that, while in a universe without extinction the mean absolute magnitudes are the same for all apparent magnitudes, the presence of extinction causes the intrinsically fainter stars to become on the average also apparently fainter.

The numbers shown in Table VIII. enable us to find the distribution with regard to distance for stars brighter than a given magnitude. We are thus in a position to compare the "frequency law" resulting from the present theory with that derived by Dyson and Eddington from the proper motions.

The law found by these investigators is given by

$$2k^2re^{-k^2r^2}dr \quad . \quad . \quad . \quad . \quad (11)$$

which expresses the fraction of the total number of stars whose distances lie between $(r \pm \frac{1}{2}dr)$. The analytical form is empirical, and chosen purely on account of its adaptability to the observed data on the one hand, and mathematical treatment on the other.

For this reason it is not to be expected that the frequency curves derived from the two different points of view should show absolute agreement. Also, before comparing the frequency curves attention may be drawn to the fact that the constant term in Kapteyn's equation (10) differs appreciably from that determined by more recent values of the parallactic motions. Comparing, for instance, the parallactic motion of the sun derived from the discussion of the Boss stars as published in *Astron. Journ.*, No. 614, p. 112, with that derived by Dyson from the Carrington stars, we find the following data:—

Boss : mean magnitude 5·7 ; $V_0 = 3''{\cdot}85$ (per century),
Carrington : ,, ,, 9·3 ; $V_0 = 1''{\cdot}71$,

V_0 representing the angular velocity of the sun at distance $\bar{\pi}^{-1}$.

The corresponding logarithms of the mean parallaxes are

$$\text{Boss}: \log \bar{\pi} = -2{\cdot}02\ ; \quad \text{Carrington}: \log \bar{\pi} = -2{\cdot}38,$$

and hence:

$$\log \bar{\pi} = -2{\cdot}00 - 0{\cdot}100(m - 5{\cdot}5).$$

Comparing with (10), we notice a small diminution of the magnitude coefficient which, however, is scarcely outside the p.e. of its determination. But there is an appreciable difference in the constant term, indicating that the unit of distance according to Kapteyn is 1·35 times that derived from more recent determinations.

This difference, however, is not relevant for our purposes. It would be a simple matter to adapt the preceding results to the more recent parallax unit by increasing the quantities $\log r$, on which the calculations are based, by the constant 0·13, and by applying corresponding corrections to the constants ν_0 and α, which would leave the relation between m and $\log N$ unaffected.

For comparing the frequency curves, the investigations of the Boss stars by Eddington and of the Carrington stars by Dyson were consulted. The limiting magnitudes (visual) are 5·8 and 10·3 respectively, which according to Table V. correspond with the photographic magnitudes 6·3 and 11·2.

In the case of the Boss stars, the percentages given by Eddington in *M.N.*, vol. lxxii. p. 384, were adopted. A few per cent. of the stars remain unaccounted for; the percentages here given are based on the actual numbers shown in Eddington's table. For the Carrington stars the percentages were computed from (11), adopting Dyson's value for k.

In the subjoined table the argument $\log r$ represents the logarithm of the radius of the sphere with centre at the sun which encloses the stated proportion of stars.

TABLE XIII.

Boss Stars.		Carrington Stars.	
log *r*.	Per cent.	log *r*.	Per cent.
1·221	2·1	1·20	0·1
1·395	5·1	1·40	0·3
1·569	9·5	1·60	0·9
1·742	17·8	1·80	2·1
1·916	32·7	2·00	5·4
2·090	53·4	2·20	12·9
2·264	74·6	2·40	29·3
2·438	89·6	2·60	58·2
2·611	96·0	2·80	88·9
2·785	98·7	3·00	99·6
2·959	100·0		

Thus, in the case of the Carrington stars, for example, 5·4 per cent. of all the stars of the catalogue are contained within a sphere of radius 100 parsecs, and 99·6 per cent. within the sphere of radius 1000 parsecs.

The difference between the limiting photographic magnitudes is very nearly 5·0. In comparing with the present theory we may therefore calculate from Table VIII. the distribution in distance for stars brighter than 6·5 and 11·5. The resulting percentages are shown in Table XIV. for the adopted value $\alpha = 0{\cdot}00305$ on the one hand, and for the assumption $\alpha = 0{\cdot}00$ on the other.

TABLE XIV.

Stars brighter than 6·5 (Ptg.). Percentages.			Stars brighter than 11·5 (Ptg.). Percentages.		
log *r*.	$\alpha = 0{\cdot}00305$.	$\alpha = 0{\cdot}0$.	log *r*.	$\alpha = 0{\cdot}00305$.	$\alpha = 0{\cdot}0$.
0·64	0·1	0·1			
0·84	0·6	0·2			
1·04	1·6	0·5			
1·24	4·0	1·3	1·24	0·1	
1·44	9·0	2·9	1·44	0·5	
1·64	18·0	5·9	1·64	1·5	0·1
1·84	32·3	11·2	1·84	4·2	0·2
2·04	51·4	19·1	2·04	10·9	0·5
2·24	72·0	29·8	2·24	24·7	1·3
2·44	88·5	42·7	2·44	46·8	2·9
2·64	97·3	56·3	2·64	73·4	5·9
2·84	99·8	69·4	2·84	93·1	11·2
3·04	100·0	80·4	3·04	99·5	19·1
3·24		88·6	3·24	100·0	29·8
3·44		94·0	3·44		42·7
3·64		97·3	3·64		56·3
3·84		98·9	3·84		69·4
4·04		99·7	4·04		80·4
4·24		100·0	4·24		88·6

The curves representing these figures are shown in fig. 2, the argument $\log r$ representing the abscissæ, and the percentages the ordinates. A comparison with the curves representing the data of Table XIII. showed that a constant correction 0·06 must be deducted from the argument $\log r$ of the latter table to bring the two tables into agreement. The limiting magnitudes of Table XIV. being 0·3 larger than those of Table XIII., we should expect the values of $\log r$ to be approximately larger by 0·03. Instead they are found to be smaller by 0·06. Hence the difference $\Delta \log r$ between Table XIII. and theory amounts to $+0{\cdot}09$ instead of 0·13 as found before from the parallactic motions.

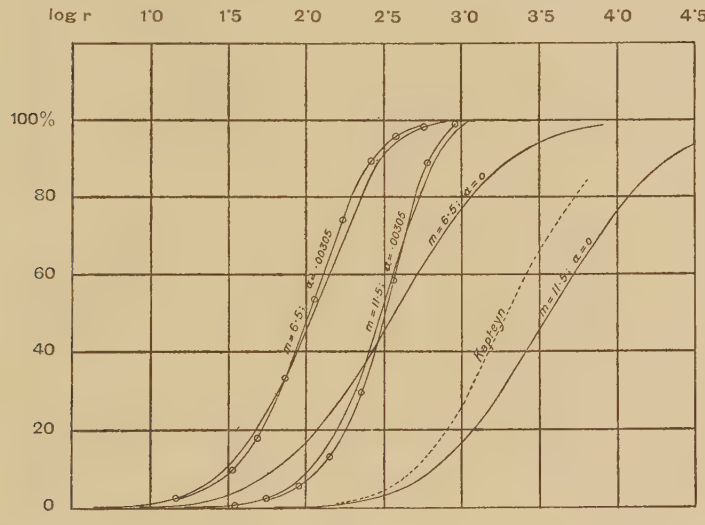


Fig. 2.

The curves representing the data of Table XIII., after applying the correction $\Delta \log r = -0{\cdot}06$, are shown in fig. 2 as the lines drawn through the small circles. These curves adapt themselves to those derived from the present theory as well as can be expected from material deduced from entirely different aspects of the problem and based on assumptions which must be considered as first approximations rather than strict analytical expressions of the facts.

The comparison with the curves derived on the assumption of the absence of space extinction reveals emphatically the enormous "shrinkage" of the visible universe due to the minute rate of extinction, on which the present theory is based.

It is further interesting to compare the frequency law here obtained with that derived from Kapteyn's data, given in table vi. on page 119 of *Astron. Journ.*, No. 566. The percentages of stars brighter than 9·5 (visual), according to Kapteyn, are shown in the dotted line curve of fig. 2, which, however, to avoid

confusion, has been shifted 0·8 unit towards the right. The slope of this curve is considerably less than that of the Carrington stars, with which it may be compared as regards the apparent magnitudes of the stars (see § 8).

Summarising the results of this section, it has been found on the assumption of a universe which fulfils the conditions

(*a*) of a star density independent of the distance;
(*b*) of a distribution of the absolute magnitudes about a mean magnitude according to the law of errors; and
(*c*) of the presence of some agency by which the absolute magnitudes are increased in proportion to the distance,

that the following facts of observation are accounted for, viz.: *first*, the statistical relation between apparent magnitudes and star numbers, based on material extending over the whole sky; *secondly*, the relation between apparent magnitudes and mean parallaxes; *thirdly*, the relation between star frequency and distance for stars brighter than a given magnitude.

§ 4. *Distribution of Stars in different Galactic Regions. Structure of the Galaxy.*

We shall now examine how far the conditions prevailing in different regions of the sky deviate from the general conditions derived on the assumption that the star density is uniform throughout the universe.

Table IV., and also the results of Chapman and Melotte and Pickering, show that for corresponding magnitudes the ratio of the star numbers, N, is approximately the same for galactic and non-galactic regions. This fact can be accounted for by assuming that in the luminosity law (7*a*)

$$M - M_0 = m - 5 \log r - \alpha r - f(\lambda, \beta),$$

where $f(\lambda, \beta)$ is a function of the angular co-ordinates λ (galactic longitude) and β (galactic latitude), but is independent of r.

Such an assumption, as Mr. Hough pointed out to me, could be explained on the hypothesis that the law of extinction is of the form:

$$\alpha r + f(\lambda, \beta).$$

The last term can be explained on the assumption that, in addition to the agency which causes the general extinction αr, the space immediately surrounding the sun is filled with absorbing matter whose density depends on the *direction*. On this assumption, the galaxy would be a purely optical phenomenon caused by a *stronger local* absorption of light in the direction of the galactic poles. The suggestion recommends itself on account of its simplicity, and deserves to be tested in the light of observations.

When the distribution of stars along the galactic circle, *i.e.*

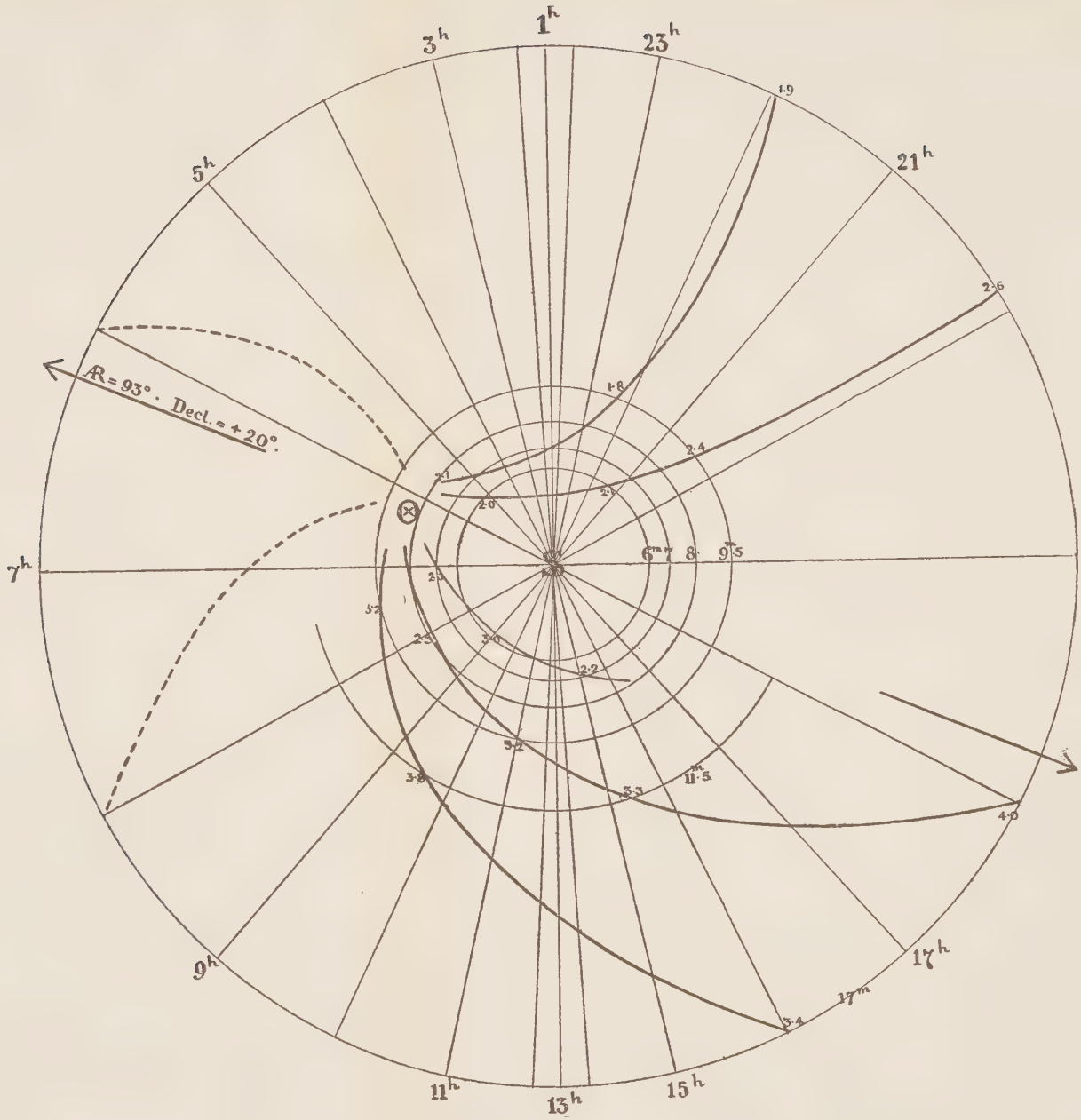

1h
3h
23h
5h
21h
7h
9h
11h
13h
15h
17h
17m
R = 93°. Decl. = + 20°.
Galactic Plane.

Fig. 3*a*.

according to galactic longitudes, is examined, the existence of strongly marked local condensations becomes at once apparent. The galaxy does not represent the appearance of a uniform ring, but rather that of an assemblage of "clusters." Stratonoff's charts (Atlas to Publications, Nos. 2 and 3) bear out this feature in both hemispheres. But if the phenomenon of the Milky Way is due to local absorption, the radial directions towards the clusters represent vistas along which this absorption is least effective. Hence the galactic longitudes of the clusters should be the same for stars of all magnitudes.

To take an example, the southern chart of Stratonoff for stars whose C.P.D. magnitude is $> 9{\cdot}6$ (mean photographic magnitude about $11{\cdot}5$) shows two well-pronounced condensations on the galactic circle at R.A. $9^h{\cdot}5$ and $15^h{\cdot}5$. The assumption of local absorption requires that similar condensations should be found on the galactic circle at the same R.A. for brighter as well as fainter stars.

For the fainter stars, table xvi. of Chapman and Melotte's paper gives the data required for ascertaining the positions of the galactic condensations. The table contains the numbers of all stars found within 225 square degrees of the Franklin-Adams plates for regions covering the whole sky. The *mean* number is 500×10^3, which corresponds with a mean magnitude $17 \pm$. In plotting the star numbers on a chart similar to those of Stratonoff, six well-defined galactic condensations in R.A. $6^h{\cdot}1$, $8^h{\cdot}0$, $16^h{\cdot}0$, $18^h{\cdot}0$, $20^h{\cdot}0$, $22^h{\cdot}8$, with relative densities $1{\cdot}6$, $2{\cdot}6$, $3{\cdot}4$, $4{\cdot}0$, $2{\cdot}6$, $1{\cdot}8$ (density of whole sky $= 1$) were located. Apparently the two condensations of the C.P.D. stars cannot be reconciled with corresponding condensations of the fainter stars on the assumption of local absorption. Similar failures are met with when the brighter stars are examined. The divergences are generally so great that the hypothesis must be abandoned.

A different view, however, seems to lead to significant results. Supposing that the luminosity law and the law of extinction are the same throughout the universe, we may assign to stars of a given apparent magnitude the mean distances shown in Table IX. (This assumption is approximately warranted, as the spread in distance for stars of given magnitude is comparatively small.) Describing round the sun in the galactic plane circles with these distances as radii and locating the condensations of stars of given magnitude on these circles, it was found possible to connect these points from one circle to another by curves which are of the nature of the wings of a spiral whose centre is situated at O (see fig. 3*a*, Plate 1). The densities within these wings, relative to the mean density, have been marked in the figure. They appear to be fairly uniform throughout the wing. [The drawn-out curves are fairly well established; the dotted curves, however, must be considered hypothetical, since only the intersections with the outer circle are known.] The material is still scanty, and definite conclusions must be deferred until more is known about

the distribution of stars between the 11th and 17th magnitudes. But the figure certainly *suggests* a structure of the galaxy in the form of a *spiral*, the sun being situated at some distance from its centre. The most noteworthy result, however, is the position of this centre as viewed from the sun. The co-ordinates of the centre of the spiral are found to be:

$$A = 93^\circ, \text{ decl.} = +20^\circ.$$

But this is almost exactly the position of the true vertex derived from the proper motions (see Eddington's *Stellar Movements*, p. 124). Such a coincidence doubtless points to a *physical* significance of the point O as a centre, not only as regards the arrangement of star matter, but also with regard to the general movements of the celestial bodies.

The same method may be applied to the arrangement of stars in a plane at right angles to the Milky Way. In table xi. of Chapman and Melotte the ratios of the star numbers found in zones of different galactic latitudes to the mean numbers (taken over the whole sky) are shown for stars brighter than specified magnitudes ranging from 5 to 17. From these numbers may be found the ratios of the numbers in each zone to the mean numbers (whole sky) between any two consecutive magnitudes m and $(m+1)$.

Drawing, therefore, concentric circles round the sun in a plane passing through the galactic poles with radii equal to the mean distances corresponding with the magnitudes $(m+\frac{1}{2})$, and mapping the ratios at the points of intersection between these circles and the radii drawn in the directions of the galactic regions I. to VIII. (Chapman and Melotte), lines of equal ratios may be constructed. These lines are shown in fig. 3*b*, Plate 2. They convey an impression of the structure of the galactic condensation in cross-section, which, however, cannot be claimed to be more than an approximate representation of the actual conditions on account of the excentric position of the sun.

Broadly speaking, the quasi-radial direction of these lines of equal density may be accepted, especially in middle latitudes and for stars brighter than 12^m. It is evident that if the arrangement of these lines were *strictly* radial, the theory of the preceding section could be rigorously applied along each radius, and the only change from one galactic region to another would be a change in the constant N_0 in (8). Hence the ratio of the numbers of bright to the numbers of faint stars would be the same in all parts of the sky, in accordance with the evidence of Table IV. and with similar results obtained by Pickering and Chapman and Melotte.

§ 5. *Relation between Proper Motion and Absolute Magnitude.*

Recent researches by Kapteyn and Adams have demonstrated the probability of a relation between proper motion and luminosity. It appears that intrinsically faint stars possess on the average larger

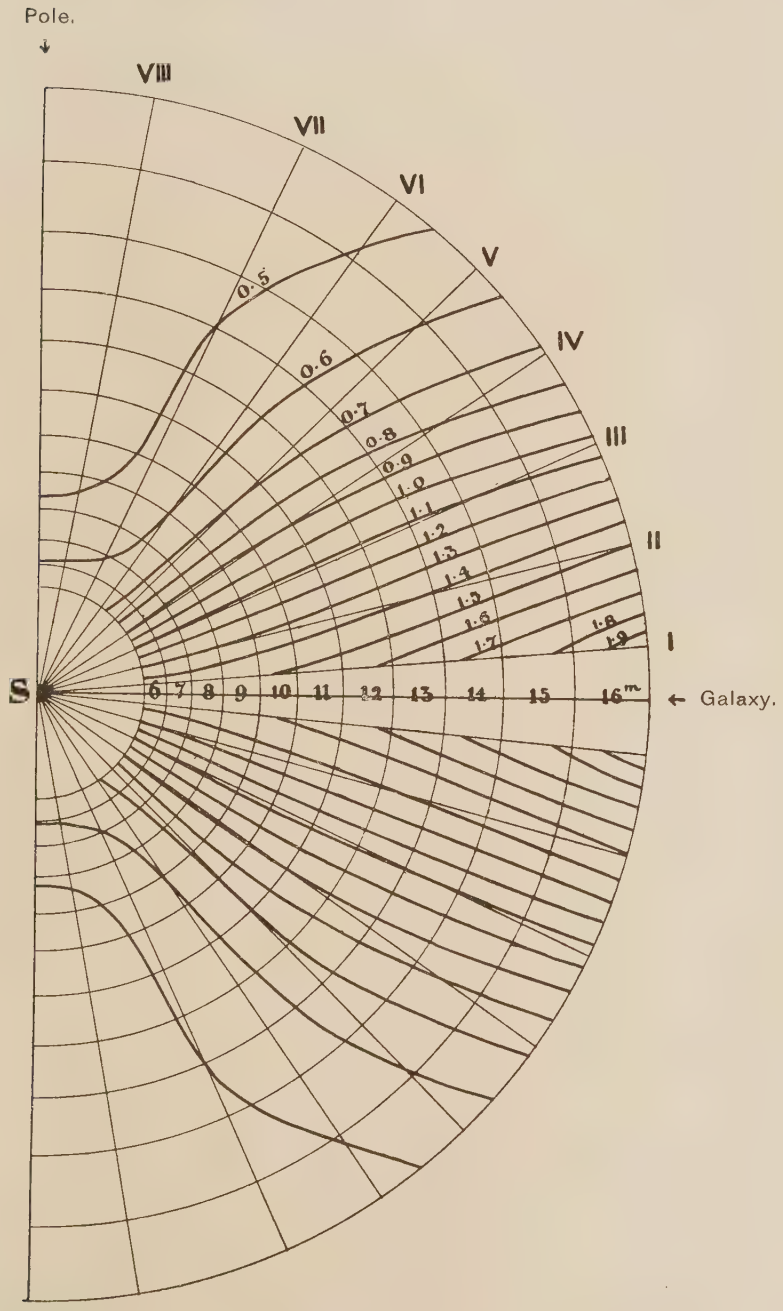

Pole.
VIII
VII
VI
V
IV
III
II
I
0·5
0·6
0·7
0·8
0·9
1·0
1·1
1·2
1·3
1·4
1·5
1·6
1·7
1·8
1·9
S
6 7 8 9 10 11 12 13 14 15 16m
← Galaxy.

Fig. 3*b*.

motions than intrinsically bright stars. The question can be examined in the light of the preceding results. It was shown that the average absolute magnitudes increase appreciably with the apparent magnitudes. Hence a corresponding increase in the average motions of the stars with apparent magnitude should be anticipated. Observations confirm this conclusion, the relation between the proper motion, μ, and the average absolute magnitude being expressed by the equation:

$$\log \bar{\mu} - \log \bar{\pi} = 0{\cdot}06\bar{M} + 1{\cdot}10 \quad . \quad . \quad . \quad (12)$$

In Table XV. columns 2 and 3 are shown for specified values of m (visual), the theoretical values of the average parallax, $\bar{\pi}$, and absolute magnitude, $\bar{M}$: the fourth column contains the values of $\bar{\mu}$ expressed in seconds of arc and computed from equation (12).

TABLE XV.

m (Vis.).	$\log \bar{\pi}$.	$\bar{M}$.	$\bar{\mu}$.	m (Vis.).	$\log \bar{\pi}$.	$\bar{M}$.	$\bar{\mu}$.
2·0	−1·37	−6·0	0″·225	8·0	−2·15	−3·8	0″·052
3·0	−1·52	−5·7	0·174	9·0	−2·26	−3·4	0·044
4·0	−1·66	−5·4	0·132	10·0	−2·36	−2·9	0·037
5·0	−1·79	−5·0	0·102	11·0	−2·45	−2·5	0·032
6·0	−1·92	−4·6	0·079	12·0	−2·54	−2·0	0·028
7·0	−2·04	−4·2	0·064	13·0	−2·62	−1·6	0·025

For comparing the theoretical values of $\bar{\mu}$ with observation, the results of Kapteyn's investigation (*Ast. Nach.* 3487, p. 110, table iii.) and for fainter stars Comstock's results (*Astron. Journ.*, No. 655, p. 55, table ix.) may be used. Table XVI. shows these results of observation in the first two columns, while the third column gives the theoretical values of $\bar{\mu}$ interpolated from Table XV.

TABLE XVI.

m.	$\bar{\mu}$ (Obs.).	$\bar{\mu}$ (Theor.).	Diff.	Authority.
2·73	0″·196	0″·190	+0″·006	Kapteyn
4·14	0·147	0·129	+0·018	,,
5·07	0·101	0·103	−0·002	,,
6·01	0·079	0·079	0·000	,,
6·93	0·066	0·065	+0·001	,,
8·5	0·045	0·048	−0·003	Comstock
11·5	0·030	0·030	0·000	,,

Equation (12) therefore represents the observed proper motions with a high degree of accuracy throughout the whole range of apparent magnitudes.

The p.m. on which Table XVI. is based have been collected in order of magnitude only, taking no account of the distances. It is

important to show that equation (12) also applies to stars selected according to observed parallaxes. The number of stars for which reliable parallaxes are available is still very scanty. For the present purpose two main sources have been utilised, viz. the results quoted by Kapteyn in Groningen Publications, No. 8, and the results of Russell's investigations (*Astron. Journ.*, No. 618–19). From these the following correlated values of $\bar{M}$ and $\log \mu/\pi$ were derived and compared with equation (12*a*).

$\bar{M}$.	$\log \mu/\pi$ (Obs.).	No. of Stars.	$\log \mu/\pi$ (Comp.).
−3·0	0·78	22	0·82
−0·4	1·01	27	0·99
+1·7	1·19	38	1·12
+4·1	1·21	29	1·26

$$\log \mu = 0{\cdot}063\bar{M} + \log \pi + 1{\cdot}01 \quad . \quad . \quad . \quad (12a)$$

The essential agreement between these two different sets of observations renders the conclusion highly probable that equation (12) expresses a general law applicable to stars under all conditions, the departures of individual stars being in accordance with the law of errors.

Considering that for any star

$$M = m + 5 \log \pi - a/\pi,$$

we may then write equation (12) in the form:

$$\log \mu = 1{\cdot}05 + 0{\cdot}06m + 1{\cdot}30 \log \pi - 0{\cdot}06a/\pi \quad . \quad (12b)$$

the constant term being the mean of the two equations (12) and (12*a*). The last term of this equation, however, is negligible within the range of observed parallaxes and proper motions. [Even at the distance of the B-stars the term a/π amounts to only 0·4 of a magnitude, and hence the last term of equation (12*b*) to only 0·024.] Neglecting, therefore, this term, we may write:

$$\log \pi_{m,\mu} = -0{\cdot}81 - 0{\cdot}046m + 0{\cdot}77 \log \mu \quad . \quad . \quad (13)$$

This equation is in close agreement with Kapteyn's empirical formula:

$$\log \pi_{m,\mu} = -0{\cdot}77 - 0{\cdot}0434m + 0{\cdot}71 \log \mu \quad . \quad . \quad (14)$$

which consequently may be considered as an alternative form of equation (12).

An important point with regard to the latter equation must, however, be considered. The proper motion, μ, is the resultant of the transverse *motus peculiaris* of the star and the parallactic displacement due to the motion of the sun. Obviously an influence of the star's luminosity on the latter component is impossible, and hence, if a connection between motion and luminosity exists, it has been introduced into equation (12) through the medium of

the peculiar motions. Now, in the case of such broad averages as are dealt with in Table XV., we may assume:

$$\bar{\mu}^2 = (V_0 \overline{\sin \lambda \,.\, \pi})^2 + \bar{\epsilon}^2,$$

where V_0 expresses the sun's motion at unit distance, λ the arc of the great circle between apex and star, and ϵ the transverse component of the *motus peculiaris*. [The value of $\overline{\sin \lambda}$ derived from Kapteyn's general discussion is 0·800 and $V_0 = 4''{\cdot}12$ per annum.]

Computing $\bar{\epsilon}$ from the values of $\bar{\mu}$ in Table XV., it is found that these quantities are closely represented by the formula:

$$\log \bar{\epsilon} = 0{\cdot}08\bar{M} + \log \bar{\pi} + 1{\cdot}12 \quad . \quad . \quad . \quad (15)$$

the only essential change in the transition from μ to ϵ being a slight increase in the factor of M from 0·06 to 0·08.

Further, assuming that the directions of the motions are distributed at random, the radial velocity, V_r, in km./sec. is expressed by

$$\bar{V}_r = \frac{9{\cdot}476}{\pi} \,.\, \bar{\epsilon},$$

and hence

$$\log \bar{V}_r = 1{\cdot}53 + 0{\cdot}08m + 0{\cdot}40 \log \bar{\pi} \quad . \quad . \quad (16)$$

and also, substituting from (13):

$$\log \bar{V}_r = 1{\cdot}20 + 0{\cdot}062m + 0{\cdot}308 \log \bar{\mu} \quad . \quad . \quad (17)$$

The equation in this form leads to an interesting comparison with observations recently made by Kapteyn and Adams (*Proc. National Academy of Sciences*, vol. i., 1915 January and July). If the radial velocities of stars whose apparent magnitudes vary within narrow limits are collected in groups in order of their proper motions, μ, then the mean apparent magnitudes of these groups may be assumed practically constant, and equation (17) reduces to

$$\log \bar{V}_r = \text{const.} + 0{\cdot}308 \log \bar{\mu} \quad . \quad . \quad . \quad (18$$

which shows that the radial velocity increases with the proper motion.

If the mean magnitude is assumed $m = 4{\cdot}5$, which is approximately that of the stars in Campbell's lists, we have from (17):

$$\log \bar{V}_r = 1{\cdot}48 + 0{\cdot}308 \log \bar{\mu},$$

from which the following values of V_r for specified values of μ are computed:

TABLE XVII.

μ.	V_r.	μ.	V_r.
″	km.	″	km.
0·008	6·8	0·30	20·8
0·02	9·0	0·40	22·8
0·05	12·0	0·50	24·4
0·10	14·9	0·60	25·8
0·20	18·4	1·00	30·2

The observed values, according to Kapteyn and Adams, are as follows:

TABLE XVIII.

Kapteyn, Type K.

μ.	V_r.	No. of Stars.	μ.	V_r.	No. of Stars.
″	km.		″	km.	
0·013	10·9	27	0·11	20·7	12
0·032	12·1	37	0·13	16·5	24
0·05	14·9	21	0·175	15·6	24
0·07	10·7	24	0·25	16·5	18
0·09	15·7	26	>0·30	26·7	19

Adams.

Type.	μ.	V_r.	No. of Stars.	μ.	V_r.	No. of Stars.
	″	km.		″	km.	
B	0·016	8·2	61	0·041	9·6	52
A	0·019	10·0	55	0·067	10·7	104
F	0·011	10·1	20	0·530	24·6	45
G	0·013	10·6	63	0·670	24·9	69
M	0·015	12·6	27	0·170	17·6	12

The comparison between the observed and computed values shows a close agreement, considering the comparatively small number of stars in the individual groups. Hence we conclude that the observed velocities of Table XVIII. may be numerically represented by the equation:

$$\log V_r = 1{\cdot}53 + 0{\cdot}08M \quad . \quad . \quad . \quad . \quad (19)$$

of which equation (18) is a special form.

An essentially different form of this equation, however, is obtained if the radial velocities are collected in order of apparent magnitude without respect to proper motions. Table XV. shows that the mean absolute magnitudes, corresponding with specified apparent magnitudes, can be well represented by the simple relation:

$$\bar{M} = -7{\cdot}0 + 0{\cdot}4m.$$

Substituting into (19) we obtain therefore:

$$\log \bar{V}_r = 0{\cdot}97 + 0{\cdot}032m \quad . \quad . \quad . \quad (20)$$

where $\bar{V}_r$ now represents the mean radial velocity of stars of a given apparent magnitude m.

Perrine (*Astrophysical Journal*, 1915 June) has examined this particular relation between $\bar{V}_r$ and m. He subdivided the stars into two groups, brighter and fainter than $3^m{\cdot}0$. The mean magnitudes of these groups are not stated, but if calculated from Campbell's lists they are found to be approximately 2·5 and 4·5. For these magnitudes equation (20) gives:

$$\bar{V}_r = 11{\cdot}2 \text{ and } 13{\cdot}0,$$

as compared with Perrine's values 11·0 and 13·6.

Equations (19) and (20) demonstrate that the usually accepted assumption of $\bar{V} = \text{constant}$ is not strictly correct. Hence if average parallaxes are determined by means of the *motus peculiares* on the assumption that $\bar{\pi}$ is proportional to $\bar{V}\bar{\pi} = \bar{\tau}$, the functional relation between such parallaxes and magnitudes must differ from that derived from the parallactic motions. The difference is seen at once if we combine equation (20) with Kapteyn's equation (11).

We then obtain:

$$\log \bar{\tau} = \log (\bar{V}\bar{\pi}) = -0{\cdot}31 - 0{\cdot}076m \quad . \quad . \quad (21)$$

i.e. an appreciably smaller coefficient of m than that shown in equation (11).

In *M.N.*, vol. lxxvi., such a determination of the mean parallaxes in relation to magnitudes based on the cross components, τ, of the Boss and Carrington stars has been published by Unthank, who finds the magnitude factor:

$$0{\cdot}072 \pm 0{\cdot}005$$

in agreement with that of equation (21).

Summarising the results of this section, it is claimed that equation (19) represents a fundamental relation between motion and luminosity. The physical cause of such a relation, however cannot be a direct one. We must conclude that luminosity acts only as the interpreter of a different agency, this agency being in all probability the *mass*.

§ 6. *Analytical Expression for the Relation between* m *and* N. *A possible Physical Explanation of the Phenomenon of Extinction.*

For stars in general, the relation between the apparent photographic magnitude, m, and the number of stars brighter than this magnitude, as established by the preceding theory, may be expressed with a degree of precision sufficient for practical purposes by the formula:

$$m = \tfrac{5}{3} \log \mathrm{N} + a\mathrm{N}^{\beta} + b \quad . \quad . \quad . \quad (23)$$

where a, β, and b represent the constants:

$$a = 0{\cdot}1954\,; \quad \beta = 0{\cdot}183\,; \quad b = -0{\cdot}75.$$

The values of m computed by this formula differ from the theoretical values of Table IX. by the following residuals:—

m.	Δm.	m.	Δm.	m.	Δm.	m.	Δm.	m.	Δm.
1·5	0·00	5·5	0·00	9·5	+0·01	13·5	0·00	17·5	0·00
2·5	0·00	6·5	0·00	10·5	+0·01	14·5	0·00	18·5	+0·02
3·5	0·00	7·5	0·00	11·5	+0·01	15·5	−0·01	19·5	+0·05
4·5	0·00	8·5	+0·01	12·5	0·00	16·5	−0·01	20·5	+0·10

The agreement down to the 18th magnitude is almost perfect, but for fainter stars the residuals become more and more pro-

nounced. Within the scope of Chapman and Melotte's investigation the formula (23) may therefore be considered a practically correct representation of the theoretical relation between m and N.

According to the seventh column of Table IX. the mean distance $\bar{r}$ is very nearly proportional to $N^{0\cdot183}$, so that the second term on the right-hand side of (23) evidently is proportional to $\bar{r}$.

It will be noticed, however, that this relation between N and $\bar{r}$, and a similar relation between N and $\bar{\pi}$, cannot be extended much beyond the range of Table IX. The changes are more clearly shown when the ratios $\frac{\Delta \log N}{\Delta \log \bar{\pi}}$ and $\frac{\Delta \log N}{\Delta \log \bar{r}}$, given in the eighth and ninth columns, are inspected.

It can be shown, from general considerations, that for infinitely small as well as infinitely large distances these ratios approach asymptotically the value 3.

When equation (23) is applied to the *statistical* relation between m and N, shown in Table III., the following numerical form is obtained:

$$m = \tfrac{5}{3} \log N + 0\cdot1946 N^{0\cdot183} - 0\cdot72 ;$$

and the residuals are:

m.	Δm.	m.	Δm.	m.	Δm.
5·0	−0·01	9·0	−0·02	13·0	−0·03
6·0	+0·02	10·0	+0·01	14·0	0·00
7·0	+0·02	11·0	−0·01	15·0	+0·04
8·0	0·00	12·0	−0·04	16·0	+0·02
				17·0	−0·02

Likewise, the visual magnitudes (Harvard), shown in table xiii. of Chapman and Melotte's paper, are represented as follows:

$$m = \tfrac{5}{3} \log N + 0\cdot1592 N^{0\cdot183} - 0\cdot90 ;$$

m.	Δm.	m.	Δm.	m.	Δm.
4·0	−0·02	8·0	0·00	12·0	+0·01
5·0	+0·01	9·0	−0·03	13·0	+0·09
6·0	+0·02	10·0	−0·04		
7·0	+0·03	11·0	−0·05		

The ratio of the factors in the second terms, taken in the sense (Photographic − Visual), is 1·22. This ratio is much smaller than would be expected on the hypothesis that extinction is due to the "scattering" of light caused by cosmic "dust." On this assumption, according to Lord Rayleigh's law, the ratio should be about 2. It is worth noticing that a comparison between the coefficient $\alpha = 0\cdot00305$ (phot.), on which the preceding calculations were based, and the coefficient $\Delta\alpha$ (Photographic − Visual) found by Jones points likewise to a small ratio, viz. 1·19. Since the

effective wave-lengths of photographic and visual light are approximately 0·43 μ and 0·52 μ, the assumption of the coefficient of absorption being inversely proportional to the effective wave-length would represent the observations.

With regard to a *physical* interpretation of extinction we are, however, by no means restricted to the acceptance of the theory of scattering. The following consideration may help in pointing out the possibility of an alternative explanation.

Let us take as an example the radiation of a black body of temperature T and at unit distance. The energy emitted, measured within a certain interval of wave-length, $d\lambda$, is given by Wien's equation:

$$dI = c_1\lambda^{-5}e^{-\frac{c_2}{\lambda T}}d\lambda \quad . \quad . \quad . \quad . \quad (24)$$

Hence the amount of energy contained between the limits $\left(\begin{matrix}\lambda_1 = \lambda - \frac{1}{2}d\lambda \\ \lambda_2 = \lambda + \frac{1}{2}d\lambda\end{matrix}\right)$ is

$$I_\lambda = \tfrac{1}{4}c_1(\lambda_2^{-4} - \lambda_1^{-4})e^{-\frac{c_2}{\lambda T}},$$

neglecting powers higher than the first in $d\lambda$. Writing this equation in logarithmic form and expressing I in terms of m_0, the apparent magnitude, we obtain:

$$m_0 = C - 2{\cdot}5 \log(\lambda_2^{-4} - \lambda_1^{-4}) + 2{\cdot}5\frac{c_2 \log e}{\lambda T} \quad . \quad . \quad (25)$$

If the body is placed at distance r, we have:

$$m_r = C + 5 \log r - 2{\cdot}5 \log(\lambda_2^{-4} - \lambda_1^{-4}) + 2{\cdot}5\frac{c_2 \log e}{\lambda T} \quad . \quad (26)$$

In these equations m_0 signifies the apparent magnitude of the body seen at unit distance, which is defined as its absolute magnitude, and denoted by the symbol M.

Provided that the range of wave-lengths, within which I has been measured, is the same, equations (25) and (26) lead to the well-known relation:

$$M = m - 5 \log r \quad . \quad . \quad . \quad . \quad (27)$$

But this conclusion applies only to the assumption that T has the same value. Supposing, however, that the radiation when leaving two bodies at different distances from the observer has exactly the same absolute temperature, are we justified in assuming that the energies on arrival are still of the same temperature? In dynamical processes the existence of dissipation of energy is recognised. Are we entitled to claim exemption for the processes involved in the propagation of light energy through space? If, on the other hand, the possibility of dissipation of radiant energy during transmission is admitted, then T must be looked upon as a function of the distance, and writing generally

$$1/T = f(r),$$

the function $f(r)$ must be of such a character that its values *increase* with the distance. Accepting as a first approximation the simplest form:

$$f(r) = d + \delta \,.\, r,$$

equation (27) has to be replaced by

$$\mathrm{M} = m - 5 \log r - 2{\cdot}5 \log e \frac{c_2 \delta}{\lambda} \,.\, r \qquad . \qquad . \quad (28)$$

which agrees with the equation on which the preceding theory is based if

$$\alpha = 2{\cdot}5 \log e \,.\, \frac{c_2 \delta}{\lambda}.$$

Hence, on the assumption that the phenomenon of extinction is due to dissipation of radiating energy, the coefficient α is found to vary inversely with the effective wave-length, in accordance with the above conclusions. At the same time, the assumption suggests that the adopted simple form of the law of extinction may not hold for very large distances where possibly higher powers of r may exert their influence.

Equation (23) is convenient when the star counts are given and the magnitudes are to be determined. Conversely, to find the star number N corresponding with a given magnitude, the equation

$$\log \mathrm{N} = 0{\cdot}219 + 0{\cdot}6154m - 0{\cdot}0104m^2 \qquad . \qquad . \quad (29)$$

may be used, its adaptability to the theoretical curve between magnitudes 3 and 18 being practically as good as that of equation (23). (See footnote on p. 274.)

Equation (29) is the empirical form adopted by Chapman and Melotte. It has been extensively used in the investigations of Kapteyn, Schwarzschild, and others, and its theoretical importance will be pointed out in subsequent investigations of this paper.

§ 7. *Comparison with the Theoretical Results of Schwarzschild.*

In *Astr. Nach.* 4557 Schwarzschild has shown that the fundamental equations of stellar statistics can be derived from the assumption of three distribution functions for the density $\mathrm{D}(r)$, the luminosity I, and the velocity V of the form:

$$\left.\begin{array}{ll} \mathrm{D}(r) = 10^{a_0 - a_1\rho - a_2\rho^2} & (\rho = -5 \log r) \\ \phi(\mathrm{I}) = 10^{b_0 - b_1\mathrm{M} - b_2\mathrm{M}^2} & (\mathrm{M} = -2{\cdot}5 \log \mathrm{I}) \\ \psi(\mathrm{V}) = 10^{c_0 - c_1\mathrm{G} - c_2\mathrm{G}^2} & (\mathrm{G} = -5 \log \mathrm{V}) \end{array}\right\} \qquad . \quad (30)$$

$$\mathrm{M} = m - 5 \log r = m + \rho.$$

Of these the second is common with the present theory, and the third related to the second by means of the equation:

$$\log \mathrm{V} = 1{\cdot}53 + 0{\cdot}08\mathrm{M},$$

which has been shown in § 5 to be in accordance with observed facts.

The present theory, therefore, differs principally from that of Schwarzschild only in the assumption regarding $D(r)$. It assumes $D(r) = \nu_0 =$ constant, but instead of

$$M = m - 5 \log r,$$

the relation

$$M = m - 5 \log r - \alpha r$$

is introduced.

Let us define a parameter r_1 by the equation:

$$\left.\begin{aligned} \log r_1 &= \log r + \tfrac{1}{5}\alpha r \\ \rho_1 &= -5 \log r_1 \end{aligned}\right\} \quad . \quad . \quad . \quad (31)$$

The volume contained within a spherical shell between the radii $(r \pm \frac{1}{2}dr)$ is $4\pi r^2 dr$, and, according to the present theory, the total number of stars contained in this shell: $4\pi\nu_0 r^2 dr$. If a function $D(r_1)$ is defined by the equation:

$$D(r_1) = \nu_0 \frac{r^2 dr}{r_1^2 dr_1} \quad . \quad . \quad . \quad . \quad (32)$$

the total number of stars in the shell may also be expressed by

$$4\pi r_1^2 dr_1 . D(r_1),$$

while the absolute magnitude is given by the relation:

$$M = m - 5 \log r_1 = m + \rho_1.$$

The form of $D(r_1)$ may be empirically determined from (32). It was found to be approximately represented by the expression:

$$D(r_1) = 10^{\alpha_1 - \alpha_1\rho_1 - \alpha_2\rho_1^2}. \quad . \quad . \quad . \quad (33)$$

which satisfies relation (32) well except for small distances. (See equation II. and Table XXI.)

Hence the present theory may be brought into formal agreement with Schwarzschild's assumptions (30), if r is replaced by r_1. This change alters considerably the results with regard to the distribution of stars of given magnitude in *distance*, but does not affect the distribution of *luminosities*.

Thus Schwarzschild found that the luminosities of stars of magnitude m are distributed about a mean magnitude $\bar{M}$ according to the law of errors, where

$$\bar{M} = -7{\cdot}7 + 0{\cdot}374m.$$

On the other hand, Table IX. (fifth column) shows the mean absolute magnitude of stars of given apparent magnitude as derived from the present theory. They are satisfactorily represented by

$$\bar{M} = -7{\cdot}17 + 0{\cdot}40m,$$

in good agreement with Schwarzschild's result. Moreover, from Table VIII. we may find the percentages of the total number

of stars between magnitudes $(m \pm \frac{1}{2})$ enclosed within shells of specified bounding radii r. The stars enclosed must be intrinsically fainter than the limiting absolute magnitude:

$$M = m - 5 \log r - ar,$$

and the percentages derived from Table VIII. should conform with Schwarzschild's frequency law:

$$\int_{\infty}^{M} \phi_m(M)N = 100 \frac{h}{\sqrt{\pi}} \int_{\infty}^{M} e^{-h^2(M-\bar{M})} dM \quad . \quad . \quad (34)$$

The comparison is shown in Table XIX. for two cases, viz. $m = 6{\cdot}0$ and $m = 10{\cdot}0$.

TABLE XIX.

$m=6{\cdot}0$; $\bar{M}=-4{\cdot}67$; $h=0{\cdot}316$. Percentages.				$m=10{\cdot}0$; $\bar{M}=-3{\cdot}07$; $h=0{\cdot}330$. Percentages.			
log r.	Table VIII.	(34.)	Diff.	log r.	Table VIII.	(34.)	Diff.
3·04	100·0	100·0	0·0	3·24	100·0	100·0	0·0
2·84	99·8	99·4	+0·4	3·04	99·5	99·5	0·0
2·64	96·6	95·8	+0·8	2·84	94·8	93·5	+1·3
2·44	86·3	85·6	+0·7	2·64	77·1	75·3	+1·8
2·24	67·9	68·2	−0·3	2·44	50·2	49·5	+0·7
2·04	46·5	47·5	−1·0	2·24	26·6	26·6	0·0
1·84	27·5	28·7	−1·2	2·04	11·7	11·8	−0·1
1·64	14·4	14·8	−0·4	1·84	4·4	4·4	0·0
1·44	6·6	6·5	+0·1	1·64	1·4	1·4	0·0
1·24	2·7	2·4	+0·3	1·44	0·4	0·4	0·0
1·04	0·9	0·8	+0·1	1·24	0·1	0·1	0·0
0·84	0·3	0·3	0·0				

According to the quantities h, the "mean error" of an absolute magnitude is $\pm 2{\cdot}24$ for stars of the 6th magnitude (phot.), and $\pm 2{\cdot}14$ for stars of the 10th magnitude.

The same considerations may be applied to cases where the distance law is known for stars brighter than a limiting apparent magnitude from other sources, as for instance for the stars of Boss's catalogue. The bulk of such stars are comprised within fairly narrow limits of magnitude, and may therefore be considered typical of the conditions of stars of the mean apparent magnitude of the catalogue. The distance law of the Boss Catalogue as found by Eddington from the proper motions is shown in Table XIII. Computing the limiting absolute magnitudes corresponding with the distances given in that table by the formula:

$$M = m - 5 \log r - ar,$$

and calculating the percentages of stars fainter than these limiting magnitudes from (34), the following comparison (Table XX.) between the percentages of Table XIII. and those computed from (34) was obtained.

TABLE XX.

log r.	Percentage of Stars. Table XIII.	(34).	Diff.
1·221	2·1	1·2	+0·9
1·395	5·1	3·4	+1·7
1·569	9·5	8·7	+0·8
1·742	17·8	18·8	−1·0
1·916	32·7	34·6	−1·9
2·090	53·4	54·5	−1·1
2·264	74·6	74·4	+0·2
2·438	89·6	89·4	+0·2
2·611	96·0	97·1	−1·1
2·785	98·7	99·6	−0·9
2·959	100·0	100·0	0·0

$$\bar{M} = -5{\cdot}60\,;\quad h = 0{\cdot}363.$$

The agreement is again quite satisfactory. The "mean error" is somewhat smaller, viz. ± 1·95, and the mean magnitude brighter than would result from theory, but they are certainly of the order to be expected.

Hence the distribution in distance of stars of given apparent magnitude may be derived from the luminosity law given in equation (34).

To show with what accuracy the relation

$$\log \mathrm{D}(r_1) = a_0 - a_1\rho_1 - a_2\rho_1^{\,2}$$

adapts itself to the condition

$$\nu_0 r^2 dr = \mathrm{D}(r_1) r_1^{\,2} dr_1 \quad . \quad . \quad . \quad . \quad \text{(I.)}$$

the values $\log\left(\dfrac{r^2 dr}{r_1^{\,2} dr_1}\right)$ were calculated for specified values of r_1, and from these the best fitting constants a_0, a_1, a_2 were derived. Thus the relation was found:

$$\log \mathrm{D}(r_1) - \log \nu_0 = -0{\cdot}75 - 0{\cdot}232\rho_1 - 0{\cdot}0174\rho_1^{\,2} \quad . \quad \text{(II.)}$$

A comparison of (I.) and (II.) is shown in Table XXI.

TABLE XXI.

log r_1.	(log D (r_1) − log ν_0). I.	II.
1·0	−0·02	−0·02
1·5	−0·07	+0·01
2·0	−0·22	−0·17
2·5	−0·56	−0·57
3·0	−1·17	−1·18
3·5	−2·02	−2·02
4·0	−3·06	−3·07
4·5	−4·23	−4·34

The values derived from equation (I.) show a progressive diminution of $D(r_1)/\nu_0$ from unity at zero distance. The values (II.), on the other hand, start from $D = 0$ at zero distance, reach a maximum at about $r_1 = 30$, and thence follow the course of (I.) within the limits of the table. The discrepancy at short distances, however, is not very serious on account of the fewness of stars contributing to the statistical equations.

The density law (II.) follows also immediately from the theoretical equations of Schwarzschild, developed in his paper "Ueber die Integralgleichungen der Stellarstatistik," *A.N.* 4422. Schwarzschild showed that if the relations

$$\log \phi(M) = b_0 - b_1 M - b_2 M^2$$
$$\log N = a_0 - a_1 m - a_2 m^2$$
$$M = m - 5 \log r$$

are satisfied, the density law must be of the form:

$$\log D(r) = a_0 - a_1 \log r - a_2 (\log r)^2 .$$

But in the preceding theory the first condition was assumed, and the second was found as result * if r_1 is substituted for r. Hence the density law, when the "fictitious" distances r_1 are used, is of the form:

$$\log D(r_1) = a_0 - a_1 \log r_1 - a_2 (\log r_1)^2 .$$

With the known quantities a_1, a_2; b_1, b_2 the coefficients a_1, a_2 can be calculated from Schwarzschild's theoretical equations. (The constant term is here omitted.)

The resulting equation is found to be:

$$\log D(r_1) = \text{const.} - 0{\cdot}244\rho_1 - 0{\cdot}0178\rho_1^2 , \quad (\rho_1 = -5 \log r_1)$$

in close agreement with (II.), and representing the condition (I.) equally well.

Adopting relation (II.) and the luminosity law of the present investigation, the constants of Schwarzschild's theory are:

$$a_1 = +0{\cdot}232 \qquad b_1 = -0{\cdot}615$$
$$a_2 = +0{\cdot}0174 \qquad b_2 = +0{\cdot}0269.$$

* A comparison of the theoretical values of log N (Table IX.) with the equation

$$\log N = 0{\cdot}219 + 0{\cdot}6154m - 0{\cdot}0104m^2$$

shows the following residuals $\Delta \log N$ (Theory-Formula):

m.	$\Delta \log N$.	m.	$\Delta \log N$.	m.	$\Delta \log N$.	m.	$\Delta \log N$.
1·5	+0·04	6·5	−0·005	11·5	+0·004	16·5	−0·001
2·5	+0·016	7·5	−0·004	12·5	+0·002	17·5	+0·005
3·5	+0·002	8·5	−0·001	13·5	+0·002	18·5	+0·008
4·5	−0·002	9·5	0·000	14·5	+0·001	19·5	+0·017
5·5	−0·006	10·5	+0·002	15·5	0·000	20·5	+0·028

By means of the relations shown in the table on p. 371 of Schwarzschild's paper (*A.N.* 4557), the following equation between m and N is easily obtained, viz.

$$\log N = a_0 + 0{\cdot}590m - 0{\cdot}0106m^2,$$

in satisfactory agreement with the equation obtained from the present theory. Also, it is found that the absolute magnitudes of stars of a given apparent magnitude are distributed about a mean value

$$\bar{M} = -7{\cdot}0 + 0{\cdot}39m,$$

with a mean error $\pm 2^{m}{\cdot}21$, again in accordance with the previous result.

The relation between the mean parallax $\bar{\pi}_1$ and apparent magnitude m, according to Schwarzschild's theory, is derived from the (a, b) constants as follows:

$$\log \bar{\pi}_1 = -1{\cdot}17 - 0{\cdot}121m \quad . \quad . \quad . \quad (35)$$

and this relation should be strictly linear, *i.e.* terms of higher order than the first in m should be absent.

On the other hand, the values of $\bar{\pi}_1$ are easily derived from the present theory by consulting the star frequencies of Table VIII. In this way the following values were calculated, with which the actual mean parallaxes $\bar{\pi}$ may be compared:

m.	$\log \bar{\pi}_1$.	$\log \bar{\pi}$.
6·0	−1·922	−1·879
9·0	−2·303	−2·211
12·0	−2·654	−2·485
15·0	−3·001	−2·718
18·0	−3·366	−2·920

These values are closely represented by the formulæ:

$$\left.\begin{aligned} \log \bar{\pi}_1 &= -1{\cdot}209 - 0{\cdot}120m \\ \log \bar{\pi} &= -1{\cdot}097 - 0{\cdot}1445m + 0{\cdot}0024m^2 \end{aligned}\right\} \quad . \quad (36)$$

the first of which is in good accordance with that derived from Schwarzschild's theory.

Hence, all the results of the present theory may be derived with sufficient approximation from the mathematical form of the density and luminosity law adopted by Schwarzschild, provided that the argument for distance is taken to be r_1, where

$$\log r_1 = \log r + \tfrac{1}{5}ar.$$

It will be noticed that the change from r to r_1 has comparatively little effect on the relation between the *mean* parallaxes and apparent magnitudes. In fact, within the narrow range of magnitudes for which parallactic motions are at present available, the observations can scarcely yet decide between the two relations given in (36), although the observed curve of fig. 1 adapts itself somewhat better to the second than the first of these relations.

Judging from this criterion only, the results are therefore practically reconcilable with either the theory of extinction or that of decreasing star density.

A decisive distinction between these theories is, however, reached when the distribution of stars of a given magnitude derived from either theory is compared with corresponding results derived from the proper motions. Here Schwarzschild's theory agrees well with the distribution found by Eddington and Dyson when the argument r_1 is adopted, but fails when r is chosen. This fact must be considered as a proof in favour of the extinction theory.

§ 8. *Comparison with the Results of Kapteyn.*

The results of the preceding section have led to the conclusion that the statistical relations between m and N on the one hand, and m and $\bar{\pi}$ on the other, may be interpreted either on the assumption of uniform density and space extinction, or on the assumption of Schwarzschild's density law [equation (30)]. The sole difference lies in the interpretation of the distances. If r denotes the distance derived from the extinction theory, and r_1 that obtained from the alternative assumption, the relation between r and r_1 is expressed by

$$\log r_1 = \log r + \tfrac{1}{5}ar.$$

It is interesting to compare in this respect the distribution of stars of given magnitude derived from the present theory with that obtained by Kapteyn from the alternative assumption.

Table xxxv. of *Stellar Movements* shows the number of stars according to Kapteyn whose magnitudes are within specified limits, and which are found between specified limits of distance. Computing from these figures the percentages of stars brighter than $6^{m}\cdot5$, the numbers given in the second column of Table XXII. are found within the sphere whose radii r_1 (Kapteyn's distances) are shown in the first column. The third column contains the corresponding distances r which would result from the assumption of uniform density and space extinction.

Table XXII.

$\log r_1$.	Percentages.	$\log r$.	$\log r_1$.	Percentages.	$\log r$.
0·80	0·4	0·796	2·00	42·0	1·946
1·00	1·1	0·994	2·20	59·7	2·120
1·20	2·8	1·191	2·40	75·7	2·283
1·40	6·7	1·385	2·60	87·3	2·434
1·60	14·1	1·577	2·80	94·2	2·572
1·80	26·0	1·765	3·00	97·7	2·697

These results are graphically represented in fig. 4. The steeper drawn-out curve shows the relation between the number of stars and the distances r, whereas the more sloping dotted curve

relates the same numbers to the distances r_1, the ordinates of Table XXII. being indicated in either case by crosses. The former curve should therefore be comparable with the results of this paper. To show that this is the case, the percentages of stars brighter than 6·5 taken from Table XIV. have been inserted, the ordinates being indicated by small circles. It is seen that all these circles are almost exactly on the steeper curve.

The figure shows that the distribution of stars in distance

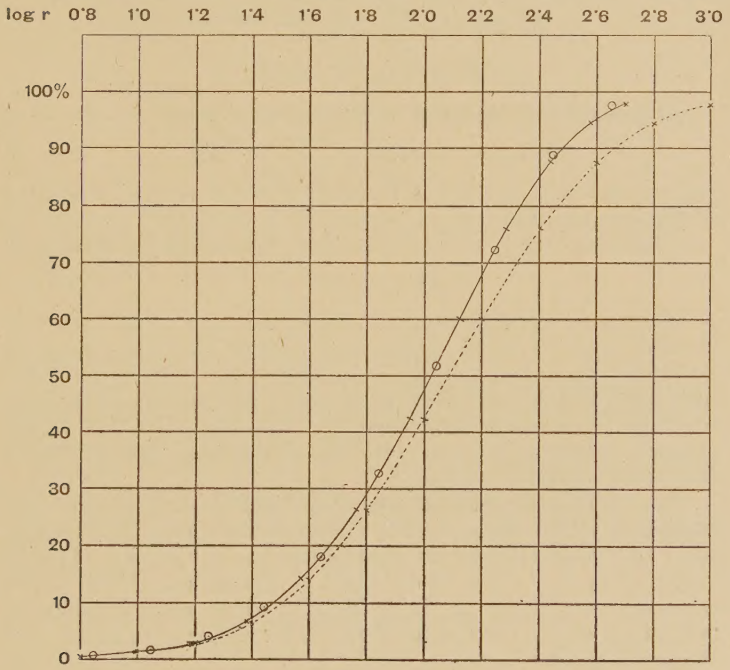


Fig. 4.

differs essentially for the two assumptions, the "spread" being less in the extinction theory. Eddington (*M.N.*, vol. lxxii. p. 384), in deducing the distance law for stars of a given magnitude (Boss's Catalogue) from the proper motions, had already come to the conclusion that the spread in distance must be *less* than that deduced from Kapteyn's data.

Kapteyn's results may be further examined from a somewhat different point of view. The figures given in the *vertical* columns of his table signify the number of stars of apparent magnitude $(m \pm \frac{1}{2})$ contained in the various shells whose limiting radii, r_1, are specified. If this number is denoted by n, and the *total* number of stars contained in the shell by N, then

$$\frac{h}{\sqrt{\pi}} \mathrm{N}\phi(\mathrm{M}) = n,$$

where

$$M = m - 5 \log \bar{r}_1,$$

$\bar{r}_1$ denoting the average distance according to Kapteyn of the stars within the shell. With the known luminosity law, $\phi(M)$, the values of N may therefore be computed for each *horizontal* column of Kapteyn's table.

The luminosity law adopted by Kapteyn is of the form :

$$\phi(I)dM = \frac{\text{const.}}{I} \times e^{-\alpha^2(\log I - T)^2} dM,$$

$$\alpha^2 = 0{\cdot}385\ ;\ T = 1{\cdot}400.$$

This expression may also be written in the form :

$$\phi(I)dM = \text{const.}\ e^{-0{\cdot}0616(M - 3{\cdot}98)^2} dM,$$

and is found to be practically identical with the expression adopted in the present work.

Denoting further by V_1 the volume of the shell, the density is :

$$D(r_1) = N/V_1 .$$

The values of log N were computed for all the vertical columns of table XXXV. (*Stellar Movements*), except the first, which is unsuitable on account of the smallness of the numbers n. For each horizontal column these values should be constant, a condition which on the whole is remarkably well fulfilled. The final values of log N subsequently used were the arithmetical means derived from each horizontal column.

We proceed to calculate the densities $D(r_1)$ and $D(r)$ on the assumptions that either r_1 or r represent the correct distances, r_1 and r being connected by the relation $\log r_1 = \log r + \frac{1}{5}ar$. (For the values of r_1 and r see Table XXII.) The results of this simple calculation are shown in Table XXIII., which gives the limiting radii of the shells, the star numbers N, the volumes V_1 and V, and the resulting densities $D(r_1)$ and $D(r)$.

TABLE XXIII.

Limiting Radii.								
log r_1.		log r.		log N.	log V_1.	log V.	log $D(r_1)$	log $D(r)$.
3·00	2·80	2·697	2·572	7·405	9·497	8·475	− 2·092	− 1·070
2·80	2·60	2·572	2·434	7·033	8·897	8·126	− 1·864	− 1·093
2·60	2·40	2·434	2·283	6·613	8·297	7·736	− 1·684	− 1·123
2·40	2·20	2·283	2·120	6·212	7·697	7·301	− 1·485	− 1·089
2·20	2·00	2·120	1·947	5·735	7·097	6·826	− 1·362	− 1·091
2·00	1·80	1·947	1·765	5·241	6·497	6·318	− 1·256	− 1·077
1·80	1·60	1·765	1·577	4·708	5·897	5·779	− 1·189	− 1·071
1·60	1·40	1·577	1·385	4·142	5·297	5·219	− 1·155	− 1·077
1·40	1·20	1·385	1·191	3·552	4·697	4·645	− 1·145	− 1·093
1·20	1·00	1·191	0·994	2·958	4·097	4·066	− 1·139	− 1·108
1·00	0·80	0·994	0·796	2·356	3·497	3·477	− 1·141	− 1·121

The assumption that r_1 represents the *real* distance leads to a density law of the form:

$$\log D(r_1) = -2{\cdot}08 - 0{\cdot}252\rho_1 - 0{\cdot}0173\rho_1^2, \quad (\rho_1 = -5\log r_1)$$

which satisfies the tabular values remarkably well (except in the case of the nearest shell), and whose coefficients are in close agreement with those derived in the preceding section.

On the other hand, the assumption that the quantities r represent the real distances leads to the density law:

$$\log D(r) = -1{\cdot}092 = \text{constant}.$$

In the former assumption the absolute magnitudes are expressed by

$$M = m - 5\log r_1,$$

and since r_1 is supposed to represent the real distance, extinction of light is not considered.

In the latter assumption:

$$M = m - 5\log r - ar,$$

and hence extinction is supposed to exist.

The first assumption leads to a density law in agreement with Schwarzschild's theory; the second assumption to a distribution in accordance with the present theory.

Finally, the first assumption leads to a distribution in distance of stars of a given magnitude which cannot be reconciled with the results derived from the proper motions; whereas the second assumption leads to a distribution in satisfactory agreement with these results.

§ 9. *Collection of Formulæ.*

Assumptions:

(*a*) The total number of stars per unit volume is the same at all distances.

Total number of stars contained within a sphere of radius 10 parsecs:

$$307{\cdot}6.$$

(*b*) Frequency of stars whose absolute magnitudes lie between $(M \pm \frac{1}{2}dM)$:

$$\phi(M)dM = 0{\cdot}1405e^{-0{\cdot}0620(M-3{\cdot}95)^2}dM,$$

$$\int_{-\infty}^{+\infty}\phi(M)dM = 1.$$

$$M = m - 5\log r_1,$$

$$\log r_1 = \log r + 0{\cdot}000610r.$$

Results derived from these assumptions :

(1) Frequency of stars of apparent magnitude, m, whose absolute magnitudes lie between $(\mathrm{M} \pm \frac{1}{2}d\mathrm{M})$:

$$\phi_m(\mathrm{M})d\mathrm{M} = 0{\cdot}1813e^{-0{\cdot}1033(\mathrm{M}-\bar{\mathrm{M}})^2}d\mathrm{M},$$

$$\bar{\mathrm{M}} = -7{\cdot}17 + 0{\cdot}40m,$$

$$\int_{-\infty}^{+\infty} \phi_m(\mathrm{M})d\mathrm{M} = 1.$$

(2) Frequency of stars of apparent magnitude, m, whose distances are between $(r \pm \frac{1}{2}dr)$:

$$\psi_m(r)dr = 0{\cdot}906e^{-2{\cdot}582(\log r_1 - 1{\cdot}434 - 0{\cdot}12m)^2}d(\log r_1),$$

$$\log r_1 = \log r + 0{\cdot}000610r,$$

$$\int_{0}^{\infty} \psi_m(r)dr = 1.$$

(3) Mean parallax of stars of apparent magnitude m :

$$\log \bar{\pi} = -1{\cdot}101 - 0{\cdot}1445m + 0{\cdot}0024m^2,$$

$$\log \bar{\pi}_1 = -1{\cdot}209 - 0{\cdot}120m.$$

(4) Number of stars brighter than magnitude m :

$$\log \mathrm{N} = 0{\cdot}219 + 0{\cdot}6154m - 0{\cdot}0104m^2.$$

The magnitudes are on the photographic scale.

To these relations may be added the provisional results of § 5, connecting proper motions with the absolute magnitudes and distances :—

(5) Mean radial velocity, $\bar{\mathrm{V}}_r$, of stars of absolute (visual) magnitude $\bar{\mathrm{M}}$:

$$\log \bar{\mathrm{V}}_r = 1{\cdot}53 + 0{\cdot}08\bar{\mathrm{M}}.$$

(6) Mean radial velocity of stars of apparent visual magnitude m :

$$\log \bar{\mathrm{V}}_r = 0{\cdot}97 + 0{\cdot}032m.$$

(7) Proper motion, $\bar{\mu}$, of stars of apparent visual magnitude m and parallax $\bar{\pi}$:

$$\log \bar{\mu} = 1{\cdot}05 + 0{\cdot}06m + 1{\cdot}30 \log \bar{\pi}$$

(neglecting the small term due to extinction).

(8) Radial velocity, $\bar{\mathrm{V}}_r$, of stars of apparent visual magnitude m and proper motion $\bar{\mu}$:

$$\log \bar{\mathrm{V}}_r = 1{\cdot}20 + 0{\cdot}062m + 0{\cdot}308 \log \bar{\mu}.$$

Royal Observatory, Cape of Good Hope :
1916 *August* 30.

Erratum in Dr. Crommelin's paper, 1916, *Nov.*

Page 44, last line of footnote, *for* 0″·05 *read* 0″·10.